DOCUMENT RESUME

ED 162 880 SE 025 381

AUTHOR Beatty, Leslie; And Others
TITLE Mathematics for the Elementary School, Grade 4, Teacher's Commentary, Part I, Unit No. 27. Revised Edition.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 62
NOTE 501p.; For related documents, see SE 025 382-384; Contains light and broken type

EDRS PRICE MF-$1.00 HC-$27.45 Plus Postage.
DESCRIPTORS Elementary Education; *Elementary School Mathematics; *Geometric Concepts; *Grade 4; *Instructional Materials; Mathematics Materials; *Number Concepts; Set Theory; *Teaching Guides
IDENTIFIERS Number Operations; *School Mathematics Study Group

ABSTRACT
This book is part 1 of a 2-part manual for teachers using SMSG text materials for grade 4. The purpose for each of 5 chapters is stated and mathematical background for the teacher is presented. Detailed lesson plans are then provided, including sequences of statements and questions, activities, and exercise sets with answers. Needed materials and vocabulary are listed. Chapter topics include: (1) concept of sets; (2) numeration; (3) addition and subtraction; (4) multiplication and division; and (5) sets of points. (MN)

Mathematics for the Elementary School, Grade 4

Teacher's Commentary, Part I

REVISED EDITION

Prepared under the supervision of the
Panel on Elementary School Mathematics
of the School Mathematics Study Group:

Leslie Beatty	Chula Vista City School District, Chula Vista, California
E. Glenadine Gibb	Iowa State Teachers College, Cedar Falls, Iowa
W. T. Guy	University of Texas
S. B. Jackson	University of Maryland
Irene Sauble	Detroit Public Schools
M. H. Stone	University of Chicago
J. F. Weaver	Boston University
R. L. Wilder	University of Michigan

New Haven and London, Yale University Press, 1963

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

CONTENTS

PREFACE

As one of its contributions to the improvement of mathematics in the schools of this country, the School Mathematics Study Group has prepared a series of sample text materials for grades 4 through 6. These are designed to illustrate a kind of mathematics curriculum that we believe appropriate for elementary schools.

This volume is a portion of these materials which were prepared by a group of 30 individuals, divided almost equally between distinguished college and university mathematicians and master elementary teachers and consultants. A strong effort has been made on the part of all to make the content of this text material mathematically sound, appropriate and teachable. Preliminary versions were used in numerous classrooms both to strengthen and to modify these judgments.

The content is designed to give the pupil a much broader concept, than has been traditionally given at this level, of what mathematics really is. There is less emphasis on rote learning and more emphasis on the construction of models and symbolic representation of ideas and relationships from which pupils can draw important mathematical generalizations.

The basic content is aimed at the development of some of the fundamental concepts of mathematics. These include ideas about:

[illegible]; numeration; the operations of arithmetic; and intuitive geometry. The simplest treatment of these ideas is introduced early. They are frequently re-examined at each succeeding level and opportunities are provided throughout the texts to explore them more fully and apply them effectively in solving problems. These basic mathematical understandings and skills are continually developed and extended throughout the entire mathematics curriculum, from grades K through 12 and beyond.

We firmly believe mathematics can and should be studied with success and enjoyment. It is our hope that these texts may greatly assist all pupils and teachers who use them to achieve this goal, and that they may experience something of the joy of discovery and accomplishment that can be realized through the study of mathematics.

Chapter 1

CONCEPT OF SETS

PURPOSE OF UNIT

[illegible] "set language." The language of sets is an effective way to express ideas, not only in mathematics but also in many other fields of knowledge. Thus, the study of sets is becoming an increasingly important part of mathematics. For this reason, [illegible] developing a concept of sets [illegible] children [illegible].

[illegible] is not new to children. For example, in [illegible] with groups of things, [illegible] the idea of sets. They [illegible] sets of objects. They may have [illegible] expressions as a set of dishes, [illegible], a set of marbles, a set of [illegible], etc. Frequently, these expressions have [illegible]. This extension of a child's knowledge [illegible] the importance of making precise statements.

[illegible] to help children in learning to [illegible] making statements that describe things, [illegible] the SMSG [illegible] grade experimental program in mathematics. It is not the intent that a theme be established [illegible] variations. However, the [illegible] in this unit will be used throughout the program whenever they are helpful to clarify meaning. In brief, the purpose of this unit is (1) to provide the children with some new language that they may use later to good advantage to express ideas precisely both orally and in writing, (2) to help the children become acquainted with one of the important concepts of mathematics, and (3) to provide the children with good understanding of some of the properties of sets and the use of this understanding to describe mathematical ideas in arithmetic and geometry more precisely.

MATHEMATICAL BACKGROUND

A set may be regarded as a collection of things. It may be a collection of pictures, of persons, of ideas, of lines, of numbers, etc. Each thing in a set is called a member or element of the set. For example, the whole numbers between 1 and 5 are 2, 3, 4. We may express this idea in this way: The set of whole numbers between 1 and 5 is {2, 3, 4}; or, if we let "the set of whole numbers between 1 and 5" be called "Set A," then we may write: Set A = {2, 3, 4}, or A = {2, 3, 4}. The members of the set are usually written within braces, { }.

Some sets contain no members at all. Such a set is called the empty set (or the null set). For example, if Set B is the set of odd whole numbers smaller than 1, Set B has no members. To show that Set B has no members, we write: Set B = { }, or B = { }.

If we wish to name a specific set, it is important that it be described explicitly or written so that there is no question about the set intended. For example, a set of children could be any set of children. However, the set of children present in your room at a certain time leaves no doubt as to the specific set. A set of numbers may be any of many possible sets; but the set of even whole numbers less than 10 refers to the specific set, {0, 2, 4, 6, 8}.

Equal Sets and Equivalent Sets. Two sets are equal if the members in one set are the same as the members in the other. It is not sufficient that the number of the members in one set be the same as the number of members in the other; the members or elements must be the same. The sets {1, 2, 3, 4, 5} and {a, e, i, o, u} have the same number of members, but they are not equal. The sets {1, 2, 3, 4, 5} and {2, 1, 4, 5, 3} are equal since their elements are the same. The order in which the elements of a set is given does not affect the set

Sometimes we make very subtle distinctions [illegible]. Consider, for example, the two sets, A and B, [illegible]

$$A = \{James, John\}$$

and B is the set of two men whose names appear in Set A. Is Set A equal to Set B? The answer depends upon what we mean by Set A. If Set A is a set of names then $A \neq B$ for B is a set of men and A is a set of names. But if Set A means the [illegible] whose names are James and John, then $A = B$.

Consider a second example. If $C = \{[illegible], [illegible]\}$ and $D = \{[illegible] + [illegible], [illegible] - [illegible]\}$, then are C and D equal? If we mean that C is a set of two symbols, namely the numerals [illegible] and [illegible], and if D [illegible] of four numerals and the symbols for addition [illegible], then C and D are not equal. [illegible] equal sets.

[illegible] between [illegible] is analogous to the [illegible] number four and the numeral four. [illegible] is meant—numeral or number. Yet, in interpreting [illegible], it may be necessary to state what is meant if the [illegible] more than one interpretation.

Two sets which are not equal but have the same number of elements are called equivalent sets. As an example, the sets $\{[illegible], [illegible], [illegible], [illegible]\}$ and $\{2, 4, 6, 8\}$ are equivalent sets but are not equal sets.

A set may be within a set. Set A is a subset of Set B if all the elements of A are also elements of B. For example, if $B = \{[illegible], [illegible], [illegible]\}$ then the subsets of B are [illegible], and the empty set { }. Any set is [illegible] the empty set is a subset of every set. [illegible] if you consider the set {Tom, Dick, Harry} [illegible]

[illegible] and Harry, [illegible] [illegible] [illegible], [illegible] "In how many ways [illegible] and [illegible] of the [illegible] to go to the ball game with [illegible]" [illegible] any one of them, or any two of them, or all three of them, or none of them. Thus, [illegible] {Tom}, {Dick}, {Harry}, {Tom, Dick}, {Tom, Harry}, {Dick, Harry}, {Tom, Dick, Harry}, and { }.

A subset of a given set is called a proper subset if the given set contains one or more elements that are not contained in the subset. [illegible] {1, 2, [illegible]} [illegible], [illegible] is not a proper subset [illegible], [illegible] {1, [illegible]}.

[illegible]

[illegible] [illegible]. If [illegible] of things, [illegible] [illegible] children in the class who are members [illegible] plan games for a class party, and suppose we let [illegible] of Set F be the children in the class who are members [illegible] for the party. If Set G = {Mary, Jane, [illegible]} and if Set F = {Ann, Dick, Susan}, then Set G [illegible] Set F will be the set of those children planning for [illegible] (assuming that these are the only [illegible]). [illegible] the union of Set G and Set F. Thus, [illegible] Set G and Set F is the set of members of both sets [illegible] {Mary, Jane, Robert, Tim, Ann, Dick, Susan}.

[illegible] write this as follows using the special symbol, $\cup$, to [illegible]: $G \cup F$ = {Mary, Jane, Robert, Tim, Ann, Dick, Susan}. [illegible] as follows: "The union of Set G and Set F is the set [illegible] Mary, Jane, Robert, Tim, Ann, Dick, and Susan." In writing the union of Set G and Set F, we always write $G \cup F$. [illegible] $G \cup F$ [illegible].

Let us consider two other sets. Set K is the set of whole numbers greater than 10 and less than 16; so, K = {11, 12, 13, 14, 15}. Let L is the set of prime numbers between 10 and 20; so, L = {11, 13, 17, 19}. The union of two sets is the ~~set~~ each of whose members is in at least one of the two sets. Thus, the union of Set K and Set L is the set, {11, 12, 13, 14, 15, 17, 19}. We can write this as: $K \cup L = \{11, 12, 13, 14, 15, 17, 19\}$. (You will notice that although 11 and 13 are members of both Set K and Set L, each of these members was listed only once in the union of the two sets.)

The union of more than two sets is defined in a similar way to the union of two sets: The union of more than two sets is the set made up of the elements that are in at least one of the given sets. For example, let $A = \{a, b, c, d\}$, $B = \{c, d, e, f\}$, and $C = \{e, f, g, h\}$. We observe that $A \cup B = \{a, b, c, d, e, f\}$ and $B \cup C = \{c, d, e, f, g, h\}$. If we write $A \cup (B \cup C)$ or $(A \cup B) \cup C$, we obtain the set $\{a, b, c, d, e, f, g, h\}$. The union of three sets A, B, and C may be found by finding the union of $(B \cup C)$ with A, i.e., $A \cup (B \cup C)$ or by finding the union of C with $(A \cup B)$: i.e., $(A \cup B) \cup C$. This is an illustration of the use of the associative property (to be met again in a later chapter). At first thought it may not seem important here but its use is necessary in finding the union of more than two sets. The operation of union is defined first for only two sets but the associative property allows us to define the union of three (or more) sets.

<u>Intersection</u> <u>of</u> <u>Sets</u>. We are familiar with the idea of <u>intersection</u> as it relates to highways. The intersection of two highways is that part of the highway common to both. We now define <u>intersection</u> as it relates to sets. Consider again Set K and Set L

$$K = \{11, 12, 13, 14, 15\}.$$

$$L = \{11, 13, 17, 19\}.$$

The intersection of these two sets is the set of members common to both sets. Specifically, the intersection of Set K and Set L is the set, {11, 13}. We can write this as follows using the

special symbol, $\cap$, to indicate intersection: $K \cap L = \{11, 13\}$. In writing intersection of Set K and Set L, we always write $K \cap L$, not Set $K \cap$ Set L. We read this as "The intersection of Set K and Set L is the set whose members are 11 and 13," or more briefly, the set K intersection L is $\{11, 13\}$.

Now let us consider Set R, the set of whole numbers 1 through 5 (this is to include both 1 and 5), and Set S, the set of whole numbers 6 through 10 (this is to include both 6 and 10).

$$\text{Set } R = \{1, 2, 3, 4, 5\}$$

$$\text{Set } S = \{6, 7, 8, 9, 10\}.$$

Set R and Set S have no members in common. Therefore, the intersection of Set R and Set S is the empty set. We write $R \cap S = \{\ \}$.

The intersection of more than two sets is defined in a way similar to the intersection of two sets. The intersection of more than two sets is the set made up of the elements that are in all of the given sets. For example, let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{3, 4, 7, 8\}$. We observe that $A \cap B = \{3, 4\}$ and $B \cap C = \{3, 4\}$. If we write $A \cap (B \cap C)$ or $(A \cap B) \cap C$ we obtain in each case the set $\{3, 4\}$. This shows that

$$A \cap (B \cap C) = (A \cap B) \cap C$$

and either $A \cap (B \cap C)$ or $(A \cap B) \cap C$ is the intersection of the three sets A, B, and C. As suggested for union, the associative property holds for the operation of <u>intersection</u> also.

TEACHING THE UNIT

THINKING ABOUT SETS

Objective: To develop the idea of a set, the ability to describe a set, and to name the members of a set

Vocabulary: set, member of a set, braces, { }, empty set

Teaching Procedures:

In mathematics we learn about numbers and things we can do with numbers. Also, we want to learn more about figures like these:

> Draw on the board (or refer to models in the room) pictures of a triangle, square, intersecting lines, point, circle, angle. Bring out idea that we want to learn both about arithmetic and about geometry.

There are some new ideas in mathematics that help us in thinking about numbers and about figures like these we have on the board (or in the room). One of these ideas is the idea of set.

> Talk about idea of set and how we use it everyday. The following may serve as a starting point. Include other illustrations.

We talk about the set of dishes in our cafeteria. We can describe the chairs in our room as the set of chairs in our room. There is the set of numbers on a thermometer, on a ruler, and on a calendar for a particular month. We might say: Name the set of numbers on the face of a clock. What numbers would we name?

> Have children name sets of things. When children name sets of things, be certain that they are explicit in naming the set such that only one specific set is appropriate. For example, a set of numbers is not a specific set but the set beginning with 10 and counting by two's to 20 is a specific set.

Likewise, a set of chairs may be the set of chairs in a certain classroom, in the library, in the store, in a particular home, or even a particular room in a particular home, etc., but the set of chairs in your classroom is a specific set. You might wish to use such ideas as the set of children in your fourth grade, the set of children in all fourth grades in your school, the set of children in all fourth grades in your state, the set of children in your room who are nine years old, the set of girls in your school, the set of boys in Cub Scouts, etc. Also, the set of numbers the children would use in counting 10 objects: the set of numbers between 0 and 50 starting at 1 and counting by two's: the set of numbers obtained in starting with 1 and counting indefinitely, noting that there is no last one of these. In identifying these sets, use illustrations where there are many in the set as well as few in the set, or even just one in the set.

When we talk about different sets of things, we can often name the things in a set. For example, if we name the set of children in our class whose names begin with S, we can write: {Sandra, Steve, Susan}.

Use names fitting to your class. Name some other sets and list the members of the set. Bring out the idea that they can be written in different ways but that usually, in mathematics, we write them using braces. (Since braces, sometimes called curly brackets but we shall use braces, may be difficult for some children to make you may permit them to use brackets, [ ]. Do not let them use parentheses, ( ).) Illustrate: The set of children in our room whose names begin with S: {Sandra, Steve, Susan}: the set of vowels: {a, e, i, o, u}. Name other sets and list the members of each set.

In our study of sets we will call _each_ of the things of a set a _member_ of the set. For the set of children whose names begin with S, Sandra is a member, Steve is a member, and Susan is a member. In the set of the five numbers {1, 2, 3, 4, 5} each of these numbers is a member of the set. Sometimes a set has no members.

> Ask questions about the members belonging to the set of elephants in your room the set of letters between a and b, the set of men in your town (city) who are 10 feet tall, the set of two place numerals which name numbers less than 10, the set of letters used to spell _cat_ that are also used to spell _dog_, etc. Bring out the idea that there are no members for each of these sets. Use illustrations suitable to interests of children in your particular class.

For all of these sets, we have said there are no members.

> You may wish to explore with children possible names for this set. In any event, bring out the idea that the name of a set that has no members is the _empty set_. You may wish to identify some empty sets within the room, the school, the community, etc.

We have a special way of writing the empty set. Suppose Set K is the set of elephants in our classroom. If there are no elephants in our classroom, we can write:

K = { }.

See next page--THINKING ABOUT SETS as in Pupils' Book, page 1.

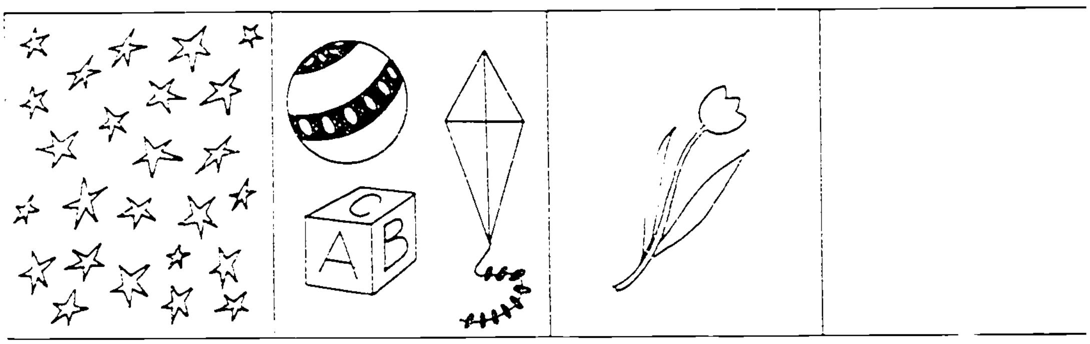
Note. Feel free to provide more practice material if needed for any section of the unit. You will find some supplementary exercises for each section at the end of the unit.

Chapter 1

CONCEPT OF SETS

THINKING ABOUT SETS

These are pictures of sets.



A Set of Stars A Set of Toys A Set of Flowers An Empty Set

You can think of many sets of things--

The set of children in your school;

The set of children in your class;

The set of numbers, 1, 2, 3, 4, 5, and so on;

The set of numbers, 1, 3, 5, 7, 9, 11, and so on;

The set of numbers, 2, 4, 6, 8, 10, 12, and so on;

The set of letters in the alphabet;

The set of boys in your class who are ten feet tall.

A set is a collection of things. Some of these collections can be sets of objects, sets of people, sets of pictures, and sets of numbers. Think of some examples of sets of things.

A thing that belongs to a set is a <u>member</u> of that set. Each of the letters, b, r, s, t, y, is a member of the set of letters in our alphabet. You are a member of the set of children in your school.

There are sets that have only <u>one member</u>. The set of letters in our alphabet between d and f has only one member. It is the letter e.

There are sets that have <u>no members</u>. The set of children in your class, who are less than four years old, has no members. If a set has no members, it is called the <u>empty set</u>.

We use capital letters for <u>names of sets</u>.
You may use any capital letter you wish.
The letter you choose may help you remember the set.
The states New York and California are members of the set of states of the U. S. A. We may call this set, Set C. We write

C = {New York, California}

The counting numbers between 4 and 8 are 5, 6, 7.
We may call this set, Set N. We write:

N = {5, 6, 7}

Exercise Set 1

List the members of each set:

1. The first five letters of the alphabet. {a, b, c, d, e}
2. The numbers that you use when you count the first five children in your classroom {1, 2, 3, 4, 5}
3. The numbers counting by 2's, beginning with 1 and ending with 9. {1, 3, 5, 7, 9}
4. The numbers counting by 2's, beginning with 6 and ending with 16. {6, 8, 10, 12, 14, 16}
5. The letters in your first name.

   (A letter may appear in your first name more than once. List it only once in the set.) Answers will vary.

6. The days of the week whose names begin with "M".

   {Monday}

7. [illegible] in your class less than six years old.

   { }

8. The names of the months whose names begin with letter J.

   {January, June, July}

9. The numbers between [illegible] that are larger than [illegible].

   { }

10. BRAINTEASER: The letters which are in the [illegible] of your [illegible] and not in your last name. Answers will vary.

NUMBERS

Objective: To help the pupil see the difference between the set of counting numbers and the set of whole numbers as defined; also, to identify within the set of whole numbers the set of even numbers and the set of odd numbers

To associate with each set of things the number of members of the set

Teaching Procedures:

Follow the suggestions given in the pupil text. Counting by 2's beginning with 0 and then with 1 will name the members of the set of even numbers and the set of odd numbers. Note the difference between the set of counting numbers and the set of whole numbers as used here.

Determining the number of members of the set is not a new experience, but children should have opportunity to do this at this time.

NUMBERS

When you first learned to count, you began with 1. You counted 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and so on. You can count much farther than 12 now. No matter how far you can count, there are still more numbers. If you knew how to count them, you could keep on counting as long as you live. Then, there would still be more numbers. These numbers used in counting are called counting numbers.

In arithmetic there is a set of numbers called the set of whole numbers. These numbers are 0, 1, 2, 3, 4, 5, 6, and so on. We may write the set of whole numbers this way:

W = {0, 1, 2, 3, 4, 5, 6, . . .}

We may write the set of counting numbers this way:

C = {1, 2, 3, 4, 5, 6, . . .}

We cannot write all the whole numbers. We use the three dots, . . . to mean that there are more numbers than we can write.
The number 0 is the first one written in Set W.
The number 6 is the last number written in the Set W.
But, the number 6 is not the last whole number.
It is just the last number written in Set W.

We write:

W = {0, 1, 2, 3, ...}

W = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...}

We have used two different ways to name the same set.

[illegible]

[illegible] 11, 13, 15, 17, 19, 21, 23, . . .

[illegible]

[illegible]

[illegible] empty set.

[illegible] empty set [illegible]

Exercise Set 2

[illegible] 1. 2. 3. 4. and 5. Below each set [illegible]

[illegible]

[illegible] 5

[illegible] 2

[illegible]

{Bobby, George, Laura}

{3, 4, 5, 6, 7}

{a, e, i, o, u}

{potato, celery, bacon, candy}

{ }

SETS WITHIN SETS

[illegible] within a given set, [illegible] diagrams to [illegible] in another

[illegible]

[illegible]:

Sometimes [illegible] a set. [illegible] talk about the set [illegible] in our class. Within this set is a set of boys. If [illegible] all the names we can get a picture of this [illegible] this:

[illegible]

[illegible] 1 through 10. Within [illegible] numbers.

[illegible]

[illegible] 1 through 10, is
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
[illegible] numbers, 1 through 10 is
{1, 3, 5, 7, 9}
[illegible] numbers, [illegible] 10, is
{2, 4, 6, 8, 10}

of helping the pupils see that within a

[illegible]

[illegible]

[illegible] in a piggy-bank.

[illegible] the table.

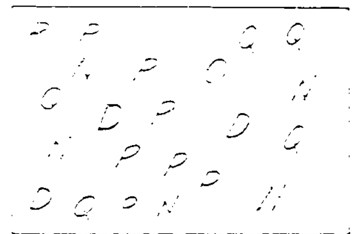
[illegible] they fall.

[illegible] nickel.

[illegible] penny.

[illegible] dime.

[illegible] quarter.

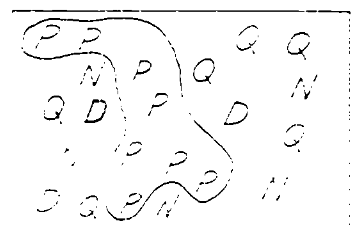


In the [illegible] there is a set of pennies.

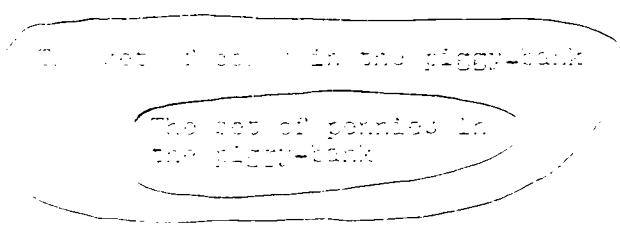
A fence is around all the pennies.

All pennies are inside the fence.

All other coins are outside the fence.



[illegible] that the set of pennies is [illegible] in the set of coins. [illegible] show it like this.

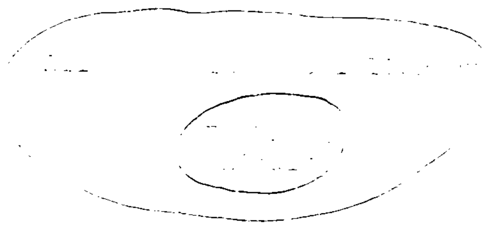


[illegible] the pennies in the picture?

They are inside the small ring.

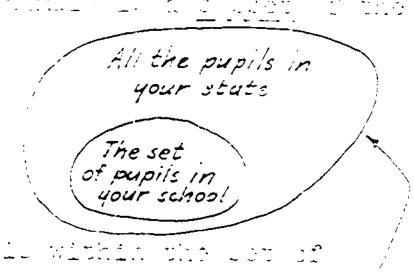
Where are the other coins that are not pennies?

They are outside the small ring but inside the big ring.



[illegible] set within a set. The set of books [illegible] is within the set of [illegible] library.

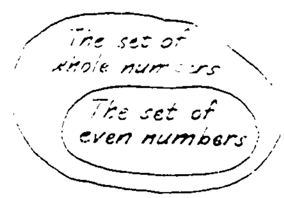
[illegible] is a subset of the [illegible] library.



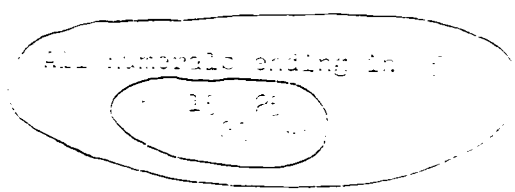
Exercise Set 3

1. The set of all pupils in your school is within the set of all pupils in your state. Draw a picture to show this idea.

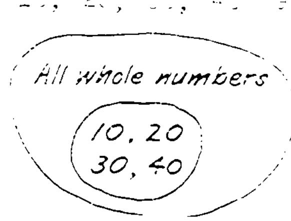
2. [illegible] that the set of all even numbers is [illegible] whole numbers.



3. [illegible] that the set of numerals [illegible], 11, 21, [illegible] is within the set of all numerals ending in [illegible].



[illegible] that the numbers 10, 20, 30, 40 are within the set of all whole numbers.



[illegible]

[illegible], Karen,

[illegible]

[illegible]

[illegible]

[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	black	[illegible]
[illegible]	[illegible]	black	[illegible]
[illegible]	[illegible]	[illegible]	10
[illegible]	[illegible]	[illegible]	[illegible]

[illegible] of the set of girls who are [illegible] years

Martha, Marion

[illegible] this set, Set B. Is Set B in Set C? Yes

Karen

[illegible] of the set of girls who have gray eyes.

[illegible] Is Set [illegible] in Set C? Yes

[illegible] of the set of girls who have black hair

Karen, Kathy

[illegible] Call this set, Set D. Is Set

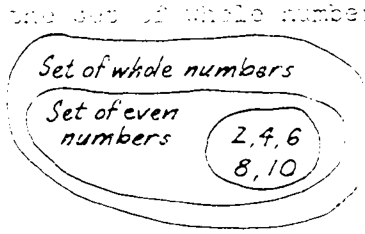
[illegible] Yes

[illegible] Yes

[illegible]: Make a drawing to show that the numbers 2, [illegible],

[illegible] are in both the set of even numbers and the even

[illegible] within the set of whole numbers.



33

EQUAL SETS

Objective: To define equal sets, to give examples of equal sets and sets that are not equal, and to determine if given sets are equal or are not equal

Vocabulary: equal set, $\neq$ meaning "is not equal to"

Teaching Procedures:

> We use a capital letter to name a set if the members of the set are written within braces. Otherwise in referring to a set in which the members are not written within braces, we shall use the word set and write for example, Set C = Set D.

We know that a set may have different names. When we say Set A = Set B, we mean that "Set A" and "Set B" are two names for the same set. If

A is {1, 2, 3} and
B is {3, 1, 2}

then Set A = Set B. The names, Set A and Set B, are just different names for the same set. The members of the sets do not need to be written in the same order. Sets with the same members are equal sets.

What is the fourth letter of the alphabet? The fifth letter? The sixth letter? Now, let us write the set whose members are the fourth, fifth and sixth letters of the alphabet. We write {d, e, f}. Call it Set A.

> Write A = {d, e, f}.

Now let us write the set of letters in the word "fed." Call this Set B. Is Set A equal to Set B? Why? Must the letters in the two sets be written in the same order for the sets to be equal?

> Have the pupils give orally or write another set which is equal to Set A. They might call it Set C and write Set C = Set A. Have the pupils find other examples of equal sets. The example below given to your pupils may suggest other examples they can give.

Write the members of the set before making the decision. That is,

K = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
L = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

The members of Set K are the numerals which name the first twelve counting numbers. The members of Set L are the numerals which mark the hours on the clock. Set K = Set L since the members of the two sets are the same.

After practice in recognizing equal sets there should be examples of sets that are not equal. If X = {1, 2, 3, 4, 5} and Y = {a, e, i, o, u} then show that these sets are not equal sets since their members are not the same. Write that Set X is not equal to Set Y by writing

Set X ≠ Set Y

Read this: Set X is not equal to Set Y. Explain that this symbol ≠ means "is not equal to". Also, we need to bring out the idea that two sets may have some members which are the same but the sets are still unequal. For example, if

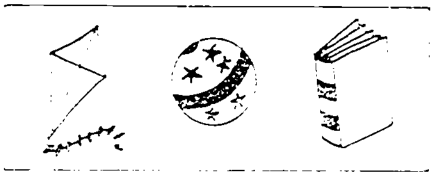
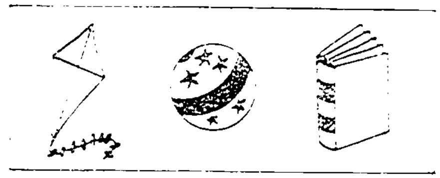
S = {12, 3, 4, 5}
T = {1, 2, 3, 4, 5}

then Set S is not equal to Set T. So write Set S ≠ Set T. Give other illustrations, describing the sets and letting the pupils determine whether the sets are equal: e. g.,

A = {ball, doll, train}	C = {1, 3, 5, 7}
B = {ball, bat, chair}	D = {3, 7, 1, 5}
Set A ≠ Set B	Set C = Set D

See next page--EQUAL SETS as in Pupils' Book, page 11.

[illegible]

[illegible]

A = {apple, pencil}

B = {pencil, apple}

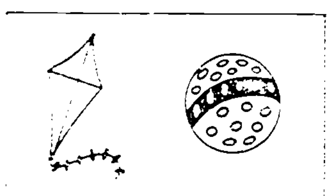
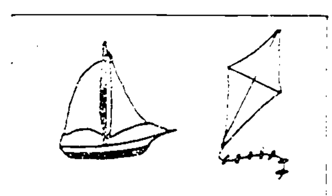
[illegible] Set B. [illegible]: Set A = Set B

M = {5, 1, 3}

N = {1, 3, 5}

[illegible] Set N = Set M? [illegible]

Yes. They have the same members.

[illegible] pictures.

[illegible] the same members.

[illegible] the number of members.

G = {apple, pencil, [illegible]}

H = {dog, car, hat}

Set G is not equal to Set H. We write: Set G ≠ Set H

F = {0, 1, 2, [illegible]}

P = {1, 2, 3}

Set F is not equal to Set P. We write: Set F ≠ Set P.

Exercise Set [illegible]

A = {4, 5, 7} B = {5, 4, 7} C = {7, 4, 5}

1. Does Set A = Set B? Yes
2. Does Set C = Set A? Yes
3. Does Set [illegible] = Set B? Yes

X = {b, a, c, d} Y = {b, c, d, a} Z = {[illegible], c, b, a}

4. Does Set X = Set Y? Yes
5. Does Set Z = Set X? No
6. Does Set Z = Set Y? No

R = {[illegible]} S = {11, 7, [illegible]} T = {8, 9, 10}

7. Does Set R = Set S? Yes
8. Does Set T = Set R? Yes
9. Does Set T = Set S? No

Here are some sets. Use these to answer questions 10, 11, 12, 13.

A = {3, 5, 7, 4, 2}

B = {2, 3, 4, 5, 6}

C = {2, 3, 4, 5, 7}

D = {6, 5, 4, 3, 2}

E = {[illegible], 7, 2, 4}

F = {2, 4, 6, 3, 5}

10. Set A is equal to what sets? *Set C and Set E*

11. Set C is equal to what sets? *Set A and Set E*

12. Are Set C, Set E, and Set A equal sets? *Yes*

13. Which sets are equal to Set D? *Set B and Set F*

14. N = {b, a, n, d}.

    Think of a set that has the same number of members as Set N but is not equal to Set N.

    Call it Set Z.

    Copy and finish: Z = { }. *Answers will vary*

15. B = {3, 7, 9, 5}

    Set E is the set of all odd numbers less than 10.

    Which is correct? Set B = Set E or Set B ≠ Set E.

    Set B ≠ Set E

16. Set A is the set of all whole numbers greater than 5 but less than 10.

    Set B is equal to Set A.

    Name the members of Set B. *6, 7, 8, 9*

[illegible]

[illegible] D = {[illegible]}

E = {[illegible]}

Which is correct? Set D ≠ Set E or Set D = Set E.

Set D = Set E

THE UNION OF SETS

Objective: To develop the understanding of the union of two sets; part of this understanding is the recognition that a member which is in both of the given sets appears only once in the union of the two sets; also to use the symbol, $\cup$, for union

Materials: Two plastic containers, a knife, key, pencil, eraser, marble, stone; other things may be used

Vocabulary: union and the symbol $\cup$

Teaching Procedures:

> Place a clear plastic container on each side of the top of your desk. Have a knife and a key in one container; have a pencil, an eraser, a marble, and a stone in the other container. Other materials may be substituted for the ones suggested as long as the major mathematical idea is made clear. Or, you may prefer to use objects on a flannel board.

Sometimes we put two sets together. This forms a new set. Let's pretend that Jack had a knife and a key in one of his pockets. We will call this Set A.

> Point to the proper container. You may wish to put a label, Set A, beside the container. Write on the chalkboard:
>
> A = {knife, key}

In another one of his pockets Jack had a pencil, an eraser, a marble, and a stone. We will call this Set B.

> Again, point to the proper container. You may wish to label this Set B.
>
> B = {pencil, eraser, marble, stone}

Jack took the things out of both pockets and put them all together on a table.

> Put all the objects together on the desk.

[illegible] We have a new set. [illegible] may call this Set [illegible]. What are the members of this new set, Set [illegible]?

> [illegible] the members of Set [illegible] on the chalkboard in whatever order the pupils give them to you: [illegible] knife, key, pencil, eraser, marble [illegible] stone.

[illegible] all the members from [illegible]? Does Set [illegible] [illegible] from Set F? We have a special way of [illegible] is the union of [illegible] and Set F. [illegible]

> [illegible] these two sets on the chalkboard:
>
> [illegible] Carol, Helen, Mary}.
> [illegible] Andy, Carol, Mary, Paul, Sam}.
>
> Describe each set in a suitable way. For example: Set X may be the set of pupils who were absent yesterday, and Set Y may be the set of pupils who were absent today. Make clear that Carol and Mary in Set X are the same Carol and Mary that are members of Set Y.

[illegible] Set X and Set Y together. Each member should be [illegible]. This forms a new set made up of all the members [illegible] Set X and [illegible]. We may call this new set, Set Z. [illegible] the members of Set Z?

> Write the members of Set Z on the chalkboard in whatever order children give them to you. For example: Z [illegible] Carol, Helen, Mary, Andy, Paul, Sam}. Be sure to have children [illegible] Carol and Mary appeared just once in the new set [illegible].

[illegible] all the members from Set Y? Does Set Z [illegible] members from Set Y? Are there any members in [illegible] in either Set X or Set Y? Is Set Z the union [illegible]?

> [illegible] what is in the chart. It may be [illegible] like this.
>
> [illegible] Carol, Helen, Mary}
> [illegible] Andy, Carol, Mary, Paul, Sam}
> [illegible] Carol, Helen, Mary, Andy, Paul,
> [illegible]

[illegible] (Bob, Carol, [illegible]

[illegible]

THE UNION OF SETS

John took a hike with his mother and father.

John kept a record of the different birds his mother saw.

He kept a record of the different birds his father saw.

Set A is the set of different birds John's mother saw.

robin	crow	sparrow

Set A

Set B is the set of different birds John's father saw.

hawk	bluejay
wren	eagle

Set B

To find all the different birds John's parents saw, we put Set A and Set B together. Our set is now

robin, crow, sparrow, hawk, wren, bluejay, eagle

This set is the union of Set A and Set B.

We write:

$A \cup B$ = {robin, crow, sparrow, hawk, wren, bluejay, eagle}

We read $A \cup B$: the union of Set A and Set B.

Your class chose some committees for a party.

The committee to select the games was Set G.

The committee to buy the prizes was Set P.

G = {John, James, Helen, Susan}

P = {John, Irene, Phyllis, Samuel}

The two committees met together. What pupils attended the meeting?

$G \cup P$ = {John, James, Helen, Susan, Irene, Phyllis, Samuel}

T = (Tuesday, Wednesday, Thursday, Friday)

F = (Monday, Tuesday, Wednesday, Thursday)

Exercise Set 5

1. A = {cat, dog, cow, horse}

   B = {duck, horse, pig}

   Which one of these sets is the union of Set A and Set B?

   X = {cat, cow, dog, duck, horse, pig}

   Y = {cow, horse, duck, horse, pig}

   Z = {cat, dog, cow, hen, duck, horse, pig}

   Answer. Set X is the union of Set A and Set B. We write

   A ∪ B = X

2. R = {10, [illegible], 30, 4[illegible], 50}

   S = {60, 70, [illegible], 90, 100}

   Which one of these sets is the union of Set R and Set S?

   M = {[illegible], 9[illegible], 110, 130, 1[illegible]}

   N = {100, 9[illegible], 80, 70, 60, 50, 40, [illegible], 20, 10}

   Copy and finish: R ∪ S = N

3. G = {a, b, c, n, m, l}

   H = {[illegible] r, q, s, t}

   Which one of these sets is the union of Set G and Set H?

   M = {q, a, m, l}

   N = {a, b, c, n, m, l, q, r, s, t}

   L = {a, l, m, q, r, t, s}

   Copy and finish: G ∪ H = N

4. J = {white, blue}

   K = {red, blue}

   Copy and finish: J ∪ K = {white, blue, red}

5. V = {18, 21, 24}

   W = {15, 18, 21, 24, 27}

   Copy and finish: V ∪ W = {18, 21, 24, 15, 27}

6. X = {s, c, a, p}

   Z = {w, a, t, e, r}

   Copy and finish: X ∪ Z = {s, c, a, p, w, t, e, r}

7. Set P is the set of odd numbers between 6 and 12. Copy and finish: P = {7, 9, 11}

   Set Q is the set of odd numbers less than 7. Copy and finish: Q = {5, 3, 1} and P ∪ Q = {1, 3, 5, 7, 9, 11}

8. Set P is the set of even numbers between 90 and 100. Copy and finish: P = {92, 94, 96, 98}

   Set R is the set of whole numbers greater than 94 and less than 96.

   Copy and finish: R = {95} and P ∪ R = {92, 94, 95, 96, 98}

9. Set T is the set of whole numbers between 10 and 11. Copy and finish: T = { }

   Set W is the set of whole numbers larger than 9 and less than 11. Copy and finish: W = {10} and T ∪ W = {10}

10. Set X is the set of counting numbers between 25 and 30. Copy and finish: X = {26, 27, 28, 29}

    Set Y is the set of even numbers between 25 and 31. Copy and finish: Y = {26, 28, 30} and X ∪ Y = {26, 27, 28, 29, 30}

THE INTERSECTION OF SETS

Objective: To develop the idea that the intersection of two sets is the set whose members are in both sets: and that if two sets have no members that are the same, then the intersection of the two sets is the empty set: also, that the symbol associated with the intersection of sets is $\cap$

Vocabulary: Intersection and the symbol $\cap$

Teaching Procedures:

> In selecting illustrations for the development of these ideas, (1) choose two sets so that some but not all members are the same in both sets; (2) two sets where all members are the same; and (3) two sets where no members are the same. In the first illustration the intersection of the sets will be the set which contains some but not all members of either set. In the second illustration the intersection of the sets is either of the equal sets. In the third illustration the intersection of the sets is the empty set.
>
> You might start by selecting two sets of children in the room so that some children will be in both sets.

Let us write the names of these members of our class on the board. We will call one Set A and the other Set B.

> For example,
>
> A = {John, Charles, Mary, Phyllis, Helen, Fred}
>
> B = {Charles, Helen, Fred, Gladys, Mildred, Ted}

The children who are in both sets are Charles, Helen, and Fred. They form another set. Let us call this set, Set C. We have a special name for this set. We call it the <u>intersection</u> of Set A and Set B.

Write: The intersection of Set A and Set B is {Charles, Helen, Fred}. Bring out idea that we have a symbol we can use and show this by writing $A \cap B$ = {Charles, Helen, Fred}.

Give other illustrations such as the following:

Set X is the set of boys in our class who are in the choir.

Set Y is the set of boys in our class who are in the band.

Write the sets as

X = {Bob, James, Charles, Henry}
Y = {Bob, Henry, John, Benny}

The boys, Bob and Henry, are in both sets. The intersection of Set X and Set Y is the set {Bob, Henry}. Or $X \cap Y$ = {Bob, Henry}

Have the pupils select the members in the intersection of two other sets. Also, have them state orally and write what the intersection set is.

Now, let us suppose that E = {1, 3, 5, 7} and that D = {5, 7, 9, 11}. The numbers, 5 and 7, are members of both sets. The intersection of Set E and Set D is the set {5, 7}. We write $E \cap D$ = {5, 7}.

Let M = {2, 4, 6, 8}, N = {2, 4, 6, 8}. The numbers which are in both sets are 2, 4, 6, 8. The intersection of Set M and Set N is the set {2, 4, 6, 8}. We write $M \cap N$ = {2, 4, 6, 8}.

In the above illustration it will be quite proper, but not required, for the pupils to write (or say) that the intersection of Set M and Set N is Set M (or the intersection of Set M and Set N is Set N).

Let us select all the odd counting numbers between 0 and 11. We will call it Set S. Now we will select all the even counting numbers between 0 and 11. We will call this Set T. These two sets (Set S and Set T) have no members which are in both sets. We say the intersection of Set S and Set T is the empty set, { }. We write $S \cap T$ = { }.

Other examples of sets whose intersection is the empty set:

1) Set J is the set of children in the class who have birthdays in January, June, or July: Set M is the set of children in the class who have birthdays in March or May.

2) X = {a, , c, d}, Y = {1, 2, 3, 4}.

3) R = {7, 8, 9}, S = {10, 11, 6}.

> You may wish to use exercises like the ones below before children use exercises on Intersection of Sets given in their book.
>
> Tell what set is the intersection of the two sets that are given in each of the following pairs. Express both orally an in writing. (Might also ask some of the children to describe each of these given sets.)
>
> (1) A = {August, September, October, November}.
> B = {September, November, January}.
>
> (2) N = {Nancy, Norman, Natalie}.
> M = {Mark, Martha, Mabel}.
>
> (3) V = {a, e, i, o, u}.
> C = {d, g, b, k, l, m, n, o}.
>
> (4) K = {4, 9, 16, 25}.
> J = {2, 4, 6, 8, 10, 12}.
>
> See next page--THE INTERSECTION OF SETS as in Pupils' Book, page 19.

THE INTERSECTION OF SETS

[illegible] picture at the right.

[illegible] Central Avenue [illegible]

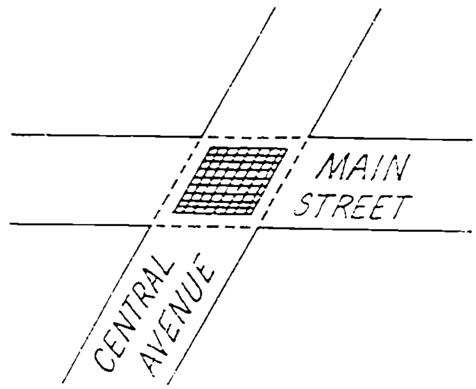
[illegible]

[illegible]

[illegible] in the picture.

[illegible]

[illegible] the <u>intersection</u> of the two streets.



[illegible] sets:

Set A	Set B
Alice	Ellen
Betty	Ken
Ken	Joe
Sue	Sue
Tom	Wendy

[illegible] members of <u>both</u> sets.

The children who are members of both sets are Ken and Sue.

This set may be written {Ken, Sue}.

This set is called the intersection of Set A and Set B.

We write: $A \cap B$ = {Ken, Sue}.

[illegible] $A \cap B$: the intersection of Set A [illegible]

[illegible] $\cap$ [illegible]

<u>Here are some more sets:</u>

[illegible] X is the set of numbers we use when we [illegible] by fives, starting with [illegible] and ending with [illegible].

X [illegible]

[illegible] Y is the set of numbers we use when we [illegible] by tens, starting with [illegible] and ending with [illegible].

Y [illegible]

[illegible] numbers that are members of both sets X and Y are [illegible], and [illegible].

The intersection of Set X and Set Y is the set {[illegible], [illegible], [illegible]}.

[illegible] write: $X \cap Y$ = {[illegible], [illegible], [illegible]}.

[illegible]

[illegible]

[illegible] is the set of even numbers less than [illegible].

[illegible] is the set of [illegible] numbers less than [illegible].

[illegible] numbers that are members of [illegible] and [illegible]

The intersection of [illegible] and [illegible] is the set [illegible]

[illegible]

Exercise Set 6

1. A = {car, train, taxi, boat}

   B = {wagon, boat, airplane, train, bicycle}

   Which one of these sets is the intersection of Set A and Set B?

   M = {car, taxi, wagon, airplane, bicycle}

   P = {boat, train}

   Q = { }

   Answer: Set P is the intersection of Set A and Set B. We write $A \cap B = P$.

2. D = {13, 17, 19, 23}

   E = {9, 11, 13, 15, 17, 19, 21}

   Which one of these sets is the intersection of Set D and Set E?

   P = {13, 17, 19, 23, 9, 11, 13, 15, 17, 19, 21}

   Q = {9, 11, [illegible], 19, 21}

   R = {17, 13}

   Copy and finish: $D \cap E =$ R

3. G = {[illegible], x, p, n, q, m}

   H = {t, [illegible], s, n, q}

   Which one of these sets is the intersection of Set G and Set H?

   I = {a, [illegible], [illegible], m, n, p, q, [illegible], s, t, x}

   J = {[illegible], s, p, q, n, m}

   K = { }

   Copy and finish: $G \cap H =$ K

4. J = {dress, shoe, hat, coat}

K = {shoe, cap, coat, dress}

Copy and finish: $J \cap K$ = {~~dress~~, shoe, coat}

5. L = {g, r, a, n, d}

M = {p, l, a, n, e}

Copy and finish: $L \cap M$ = {a, n}

6. N = {73, 59, 3, 61, 63}

O = {14, 49, 73, 46, 18, 95}

Copy and finish: $N \cap O$ = {73}

7. Set P is the set of whole numbers less than 7.

Copy and finish: P = {6, 5, 4, 3, 2, 1, 0}

Set Q is the set of whole numbers between 5 and 12.

Copy and finish: Q = {6, 7, 8, 9, 10, 11}

Copy and finish $P \cap Q$ = {6}

and $P \cup Q$ = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

8. Set R is the set of whole numbers larger than 38 and less than 44.

Copy and finish: R = {39, 40, 41, 42, 43}

Set S is the set of numbers between 36 and 46 that are not even numbers.

Copy and finish: S = {37, 39, 41, 43, 45}

Copy and finish: $R \cap S$ = {39, 4, 43}

$R \cup S$ = {39, 40, 41, 42, 43, 37, 45}

SUPPLEMENTARY PRACTICE EXERCISES

THINKING ABOUT SETS - Exercise Set _

Write the members of each of these sets.

1. The set of even numbers less than 11. {0, 2, 4, 6, 8, 10}

2. The set of counting numbers less than 20 and larger than 10. {11, 12, 13, 14, 15, 16, 17, 18, 19}

3. The set of odd numbers between 10 and 20. {11, 13, 15, 17, 19}

4. The set of whole numbers less than 17 and larger than 15. {16}

5. The set of numbers between 30 and 40 that are larger than 4 . { }

6. How many members are there in the set of letters of our alphabet? 26

7. Here is a set: {Tuesday, Thursday}. Describe this set by writing on your paper: This is a set of days in week whose names begin with T.

8. Name two sets that have no members. answers will vary

9. Make a picture of a set. Then describe the set by saying: This is a set of answers will vary

10. Describe this set in your own words:

A = {5, 10, 15, 20, 25}

Set A is the set of the first five counting numbers counting in fives

... set within Set B.

... subset ... within a set ... each of the ...

Answers will vary. Possibility {a, b, c}

... Plymouth, Rambler, Cadillac} Answers will vary. Possibility {Ford, Cadillac}

... in a year. Answers will vary. Possibility {Christmas, Thanksgiving}

... vowels. Call this set Set A. Now ... a

... {a, e, i, o, u} Answers will vary. Possibility {e, o}

... of all States in the United States. Write a

... Answers will vary. Possibility {Florida, Illinois}

... of dimes. We can say ... is a set within

... numbers.

... set of all ... numbers. Make up a subset of

... Possibility {7, 45}

P-1

[illegible] - [illegible]

1. [illegible] Set R? [illegible]

2. [illegible] = {1, ..., 11}.
[illegible] Set N [illegible] Yes

3. [illegible] Y = [illegible]
[illegible] Set Y equal Set [illegible]? Yes

4. [illegible] is the set of whole numbers greater than [illegible] and less than [illegible]. Set B is equal to Set [illegible]. Name the members of Set B. [illegible]

Here are some sets: $F = \{2, 4, 6, 8\}$
$G = \{4, 2, 8, 6\}$
$H = \{5, 3, 1, 4\}$
$K = \{8, 6, 7, 9\}$
$L = \{4, 8, 6, 2\}$
$M = \{6, 2, 8, 4\}$

5. Set F is equal to what sets? Set G and Set M.

6. Are Set H and Set L equal sets? No

7. Which sets are equal to Set K? None of them

8. [illegible] = {2, 4, [illegible] 8}. Set B is the set of all even numbers less than [illegible]. Which is correct? Set A = Set B or Set [illegible] Set B? Set A = Set B

9. Make up a set. Call this set Set Y. [illegible] make up a set that is equal to Set X. Answers will vary
Possibilities: $Y = \{0, 12, 3, 4\}$; $Y = \{1, 3, 2, 4\}$

THE UNION OF SETS - Exercise [illegible]

1. A = {1, 2, 3, 4} B = {5, 6, 7, 8}

   The union of Set A and Set B is the set: {1 2 3 4 5 6 7 8}

2. C = {b, a, n, d} D = {[illegible] k, e}

   Copy and finish: C ∪ D = {c a n d y o k e}

3. E = {5, 10, 15, 20, 25} F = {30, 35, 40, 45}

   Which one of these sets is the union of Set E and Set F?

   M = {10, 20, 25, 45, 50, 55, 60, 65, 70}

   N = {45, 40, 35, 30, 25, 20, 15, 10, 5} Set N

4. H = {2, 4, 6, 8, 10} J = {3, 6, 9, 12}

   Write the members of the union of Set H and Set J.

   H ∪ J = {2, 4, 6, 8, [illegible] 3, 12}

5. Set K is the set of odd numbers between 10 and 20.

   Set L is the set of even numbers between 10 and 20.

   Copy and finish: K ∪ L = {11, 12, 13, 14, 15, 16, 17, 18, 19}

6. Set P is the set of whole numbers between [illegible] and [illegible]

   Set Q is the set of whole numbers larger than [illegible] and less than [illegible].

   Copy and finish: P ∪ Q = {16}

7. X = {F, B} Y = {A, C, D, E}

   Copy and finish: X ∪ Y = {F a, B, E, C}

8. T = {[illegible]} V = {[illegible]}

Set [illegible] is the union of Set T and Set V. Write the members of [illegible]

[illegible]

Set K is the union of Set [illegible] and Set [illegible]. Write the members of Set K. [illegible]

9. Let M is the set of whole numbers between [illegible] and [illegible]. Let N is the set of even numbers between [illegible] and [illegible]. Copy and finish: $M \cup N = \{ \quad \}$

10. A = {[illegible], 15, 27, [illegible]} Set B has [illegible] members.

$A \cup B$ = {3, [illegible], 15, 27, [illegible]}

$A \cap B$ = {[illegible]}

B = {3 6 9 [illegible]}

THE INTERSECTION OF SETS - Exercise Set 1

1\. A = {1, 2, 3, 4, 5} B = {3, 4, 5, 6, 7}

Which one of these sets is the intersection of Set A and Set B?

X = {1, 3, 5, [illegible]}

Y = {1, 2, 4 [illegible]} *Set X*

Z = {3, 4, 5}

2\. [illegible] E = {a, c, [illegible], e}

Set F = the intersection of Set [illegible] and Set E. Write the members of Set F. *F = {a c}*

3\. R = {5, 10, 15, 20, 25} S = {10, 20, 30, 40}

Set T is the intersection of Set R and Set S. Write the members of Set T. *T = {10 20}*

4\. Set X is the set of the first five counting numbers. Set Y is the set of odd numbers between 4 and 12. Set Z is the intersection of Set X and Set Y. Write the members of Set Z.

Z = {5}

5\. K = {dogs, cats, mice} L = {pigs, dogs, cats, mice}

M = {horses, cows, dogs} Copy and finish:

$K \cap L$ = {*dogs, cats, mice*}

$K \cap M$ = {*dogs*}

$L \cap M$ = {*dogs*}

6\. H = {s, [illegible]} J = {h, a, n, d}

Copy and finish: $H \cap J$ = {*∅*}

7. A = {desk, chair, pencil} B = {eraser, chalk, chair}

C = {book, tablet, desk}

Copy and finish:

$A \cap C$ = { desk }

$A \cap B$ = { chair }

$B \cap C$ = { }

$C \cup A$ = { book, tablet, desk, chair, pencil }

$C \cup B$ = { book, tablet, desk, eraser, chalk, chair }

$B \cup A$ = { eraser, chalk, chair, desk, pencil }

8. J = {12, 18, 24, 30}. Set K has two members. One member is 6. $K \cap J$ = 18. Copy and finish: K = { 6, 18 }

9. Set R is the set of counting numbers less than 6. Copy and finish: R = { 1, 2, 3, 4, 5 }

Set T is the set of counting numbers between 5 and 11.

Copy and finish: T = { 6, 7, 8, 9, 10 }

Copy and finish:

$R \cap T$ = { }

$R \cup T$ = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }

10. Set P is the set of all even numbers between 1 and 7.

Set Q is the set of all odd numbers between 1 and 7.

Copy and finish:

$P \cap Q$ = { }

$P \cup Q$ = { 2, 3, 4, 5, 6 }

Chapter [illegible]

NUMERATION

PURPOSE OF UNIT

The purpose of this unit is two-fold: (1) to deepen pupils' understanding of principles of numeration, with emphasis upon our commonly used decimal system; and (2) to make clear the distinction between representing numbers as sets of points and representing numbers as points on a line, with consequent implications for the ideas of equality and inequality. In addition to the mathematical background which follows, one will find it helpful to study Chapter [illegible] (pages [illegible]) of _Number Systems_ (SMSG Studies in Mathematics, Volume VI).

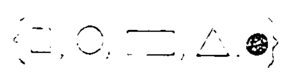
MATHEMATICAL BACKGROUND

Principles of numeration cannot be developed effectively if confusion exists regarding the terms number and numeral. These are not synonymous. A number is a concept, an abstraction. A numeral is a symbol, a name for a number. A numeration system is a numeral system (not a number system), a system for naming numbers.

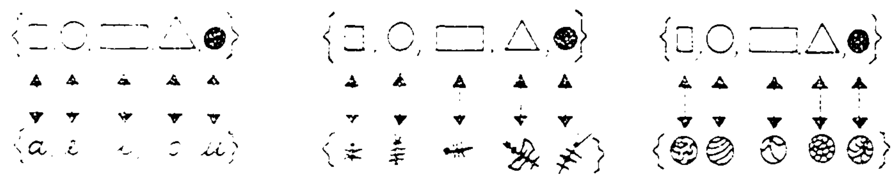
Admittedly, there are times when making the distinction between "number" and "numeral" becomes somewhat cumbersome. However, an attempt has been made in this unit to use terms such as number, numeral, and numeration with precise mathematical meaning.

We commonly represent numbers in two quite different ways: (1) in terms of sets of things, and (2) in terms of points on a line. Let us use the number 5 to illustrate each of these ways of representation.

In the first instance, the number [illegible] may be associated with a set of things, such as:

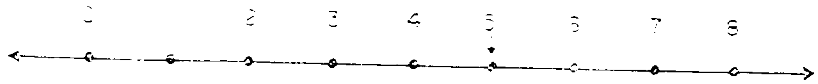


We then may associate the number 5 with any set whose members can be put in one-to-one correspondence with the members of this "model set," as shown below



All such sets possess 5 as their common number property. In this sense 5 is a cardinal number telling how many members there are in the set.

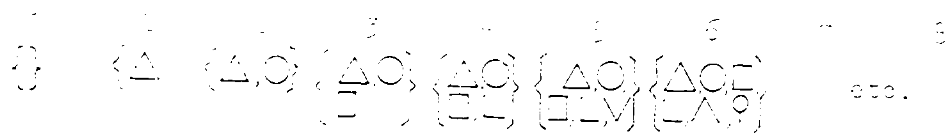
In the second instance, the number 5 may be associated with a point on a "number line" as in the picture below:



There is a <u>unique</u> point on the "number line" corresponding to each whole number; that is, there is one and only one point on the number line corresponding to a particular whole number.

Implied in the preceding discussion is the idea that the members of the set of whole numbers can be <u>ordered</u>; they can be arranged in a sequence from smaller to larger, and vice versa.

First consider the idea of order as it relates to whole numbers associated with sets of things:

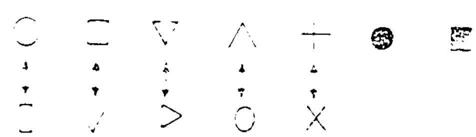


When the whole numbers are ordered in this way, we may say that:

(1) each succeeding member of the set of whole numbers is one more than its predecessor, and

(2) each member of the set of whole numbers (except 0) has a predecessor that is one less than that member.

The ideas of greater than (>) and less than (<) can be related to the matching of members of model sets. For example, consider matching the members of model sets for 7 and 5:



We say that 7 is greater than 5, (or $7 > 5$), since a matching of the members of the sets reveals an excess of members in the set whose number is 7. We also say that 5 is less than 7, (or $5 < 7$), since a matching of the members

[illegible] set [illegible] contains a [illegible] of [illegible] in the set whose [illegible] is [illegible] than [illegible] the set whose number is 7.

We consider the idea of *order* as it relates to numbers represented by points on a number line. In this instance "greater than" is interpreted to mean "to the right of," and "less than" is interpreted to mean "to the left of." Since 7 is to the right of 5 on the number line, 7 > 5. Since 5 is to the left of 7 on the number line, 5 < 7.

This may be an appropriate time to comment on the correct use of the equals sign (=). For example, when we write

$$5 + 2 = 8 - 1$$

we mean that the symbols "5 + 2" and "8 - 1" are both names for the same thing,--the number 7. In general, when we write

$$A = B$$

we do not mean that the letters or symbols "A" and "B" are the same. They very evidently are not! What we *do* mean is that the letters "A" and "B" are each being used as *names for the same thing*. That is, the equality

$$A = B$$

asserts precisely that the thing named by the symbol "A" is *identical with* the thing named by the symbol "B". The equals sign always should be used only in this sense.

Let us turn now from numbers to numerals.

The naming of numbers is a problem that has [illegible] solved [illegible] over a period of many, many years. Courses such as [illegible] the mentioned earlier (Studies in Mathematics, Volume VI) give interesting and helpful information in this connection.

For immediate purposes it will suffice to consider only [illegible] underlying nature of the scheme for naming numbers that [illegible] use commonly today.

We are so familiar with our decimal system of numeration that we sometimes fail to sense clearly that it is but an

example of a broad class of numeration systems all of which have the same feature of place-value with different bases.

Our system is called the decimal system because of the use that is made of sets of ten and the manner in which these sets of ten are built into the meaning of a numeral. (The word decimal comes from the Latin word "decem" which means "ten".) There are ten symbols which we use in writing numerals in the decimal system. These symbols for the digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The decimal system is a place-value numeration system since the size of the set represented by a digit in a numeral depends upon the position of the digit in the numeral. When we say that the numeral 213, for example, is a name for the number 2(ten times ten) + 1(ten) + 3(ones) we are recognizing the grouping in sets of ones, tens, and ten times ten (hundreds) by the positions of the digits 2, 1, and 3 in the numeral. Each successive place to the left in the numeral represents just ten times that of the preceding place and the digit in any place names the number of sets of ones, tens, hundreds, etc. for that place. Thus in any numeral in the decimal system if we read from right to left the digit in the first place names the number of ones, in the second place the number of tens, in the third place the number of ten x ten, in the fourth place the number of ten x ten x ten, and so on. Because of this grouping by sets of ten and because of the place value a digit has by virtue of its position in a numeral the decimal system of numeration is a place value numeration system with base ten. The names <u>decimal system of numeration</u> and <u>base ten system of numeration</u> may be used interchangeably. They have exactly the same meaning.

Now consider a place-value numeration system in which the base is some number other than ten, say five. We need now to think of grouping in sets of ones, sets of five, sets of five times five, and so on, just as in the decimal system where we think of grouping in sets of ones, sets of ten, sets

of ten times ten, etc. Look at the marks in the sets pictured below. If there are no marks we can name this number by 0; one mark by 1, two marks by 2, three marks by 3, four marks by 4.

Now in naming the number to indicate five marks we must write a place-value numeral which is to mean 1 set of five and no ones. This numeral is written 10_{five} in the "base five system" and is read one zero, base five. Notice the significance of place value: 0 means no sets of ones, and 1 means 1 set of five by virtue of their position. We need some way to show we are grouping in sets of five. This is done by writing five as a subscript. Observe that there is no change in the concept of the number five but only a new way of writing the numeral for it. In the base five numeration system we need only the symbols 0, 1, 2, 3, 4. When one of these is written alone, i.e., in a one place numeral it has the same meaning in the base five system as in the decimal system. Thus 1, 2, 3, and 4 written singly name the cardinal numbers for sets of one mark, two marks, three marks, and four marks respectively. Consequently if we are writing numerals of one place in the base five system the subscript "five" is unnecessary although it is not incorrect to write it.

Let us continue with counting in the base five system using sets of marks with the corresponding numeral in base five system beside the set.

XXXXX X	11_{five}	One set of five and one one
XXXXX XX	12_{five}	One set of five and two ones
XXXXX XXXXX	20_{five}	Two sets of five and no ones
XXXXX XXXXX X	21_{five}	Two sets of five and one one
XXXXX XXXXX XXXXX	30_{five}	Three sets of five and no ones
XXXXX XXXXX XXXXX X	31_{five}	Three sets of five and one one
XXXXX XXXXX XXXX XXXXX XXXX	44_{five}	Four sets of five and four ones

This is as far as we shall write in base five numerals in this chapter. For counting further and for writing numerals in other base numeration systems refer to [illegible] Mathematics for Junior High School, Volume I, Part I, Student's Text. Comparison might be made with the chart on the next page for guidance in writing base five numerals and numerals in other base numeration systems. Observe that in the base nine system only nine symbols for digits are needed, in base eight system only eight symbols are needed, and in base three system only three symbols are needed. The position of a symbol in any of the numerals and the base of the numeration system tell us the number represented by the numeral. Consider, for example, the numeral 212 found in column three in the chart on the next page. (In the chart no subscript is written for 212 since its position in the chart indicates the base is three.)

212_{three} = 2(three times three) + 1(three) + 2(ones)

It will be observed from the chart that this names the same number that is named by 23 in the base ten column. Although 212_{three} and 23 are different names for the same number it is not advisable at this stage to write $212_{three} = 23$ since this would seem to be explaining one base in terms of another. It will be better to emphasize meanings by use of such examples as the following which do not "mix" bases.

111_{four} = 1(four × four) + 1(four) + 1(one)
(Read: one, one, one, base four)
= 1(sixteen) + 1(four) + 1(one)

375_{eight} = 3(eight × eight) + 7(eights) + 5(ones)
(Read: three, seven, five, base eight)
= 3(sixty-fours) + 7(eights) + 5(ones)

435_{six} = 4(six × six) + 3(sixes) + 5(ones)
(Read: four, three, five, base six)
= 4(thirty-sixes) + 3(sixes) + 5(ones)

100_{seven} = 1(seven × seven) + 0(sevens) + 0(ones)
(Read: one, zero, zero, base seven)
= 1(forty-nine) + 0(sevens) + 0(ones)

243_n = 2(n × n) + 4(n's) + 3(ones) where n represents any number not greater than ten. Read: two four three base n.

These preceding paragraphs are not meant to suggest a procedure in presenting the subject to the pupils but to give some understanding of the meaning of place-value numeration systems with different bases. The teaching procedure is suggested in the section <u>Grouping</u> <u>In</u> <u>Base</u> <u>Five</u> and in other sections of this chapter.

Base							
Ten	Nine	Eight	Seven	Six	Five	Four	Three
[illegible]	[illegible]	[illegible]	1	[illegible]	[illegible]	[illegible]	1
[illegible]	[illegible]	2	2	2	2	2	2
[illegible]	[illegible]	[illegible]	[illegible]	3	3	3	<u>10</u>
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	<u>10</u>	11
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	<u>10</u>	11	12
[illegible]	[illegible]	[illegible]	[illegible]	<u>10</u>	11	12	20
[illegible]	[illegible]	[illegible]	<u>10</u>	[illegible]	12	13	21
[illegible]	[illegible]	<u>10</u>	[illegible]	13	[illegible]	20	22
[illegible]	<u>10</u>	[illegible]	12	[illegible]	[illegible]	21	<u>100</u>
<u>10</u>	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	22	101
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	21	23	102
[illegible]	[illegible]	[illegible]	[illegible]	20	22	30	110
[illegible]	[illegible]	[illegible]	[illegible]	21	23	31	111
[illegible]	[illegible]	[illegible]	[illegible]	22	[illegible]	32	112
[illegible]	[illegible]	[illegible]	[illegible]	23	[illegible]	33	120
[illegible]	[illegible]	[illegible]	22	[illegible]	[illegible]	<u>100</u>	121
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	101	122
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	102	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	103	201
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	110	202
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	210
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	112	211
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	113	212
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	120	220
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	<u>100</u>	121	221

[illegible] base numeral always appears [illegible] Similarly,

In a particular base system the numeral 100 always designates the base squared.

An understanding of the general scheme of positional notation will contribute to a better understanding of the base ten system of numeration in particular.

Materials for Chapter 2

1. Flannel board and cut-outs
2. Number line (0 through 100)
3. Packets of twenty to thirty objects which can be counted (A demonstration set should be large enough so it may be seen from all parts of the classroom. The pupils may have smaller objects suitable for work at their desks.)
4. Bundles of sticks, straws, or cardboard strips for use in place-value charts.
5. Abacus
6. Place-value charts may be made that will show groupings of ten, five, and three. Samples are as follows:

Hundreds	Tens	Ones

Fives	Ones

Threes	Ones

GROUPING IN BASE FIVE

Objective: To understand grouping in base five

Materials: A packet of twenty to thirty objects for each child in the class
Flannel board with cut-outs for teacher demonstration

Vocabulary: Numeration system, base ten

> In mathematics, it is important that we understand the structure of the decimal numeration system. By numeration system we mean a system of writing or stating numbers in order of their size.
> In preparation for introducing this section, the teacher may wish to have a packet of twenty to thirty objects, which can be counted, for each child in the class. This will give the child an opportunity to work with concrete materials in discovering relationships for himself.

Exploration:

Let us imagine a shepherd, living many years ago, in a large open field tending his sheep. In the beginning, the shepherd had only two or three sheep and could tell at a glance when one was missing. He did not need to know how to count. Later, his flock increased to nine and he could not tell quickly when one of his sheep was missing. He <u>had</u> to find a way of counting his sheep. There might be days when a lamb would be injured or lost. Unless the shepherd had some way of counting his sheep, he would not know there was a missing lamb. He finally learned to count his sheep by using his fingers. For each sheep he used one finger. This system worked well as long as he had no more than ten sheep.

The day came when more sheep were added to his flock. Again, he was faced with the problem of counting his sheep. He finally decided to use rocks. Each morning, after ten sheep (one for each finger) had passed through the gate, he

[illegible] rock in the pouch hanging at his side. This would tell him [illegible] as many groups of ten sheep as he has rocks in his pouch and [illegible] how many more he had shown on his fingers. For instance, if he counted twenty-four sheep he would have [illegible] rocks in his pouch. There would be four sheep left over which he would count on his fingers.

When the sheep come back in the evening, he repeats this process. He was able to tell if there were as many sheep in [illegible] as there were in the morning.

[illegible] we can use the same system. We can write a numeral representing the number of sheep without having to use rocks. When [illegible] the numeral [illegible], we know by the position of the [illegible] two sets of ten each. The position of [illegible] tells [illegible] four more sheep.

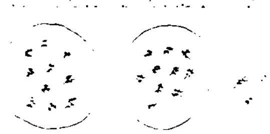
[illegible] containing nine objects. Let us [illegible] system of numeration. Count nine objects. [illegible] exactly how this is [illegible] represented. [illegible] nine objects [illegible]

[illegible] set of ten [illegible]

[illegible]

[illegible] objects [illegible] position of [illegible] it represents one set of ten [illegible] position [illegible] in this [illegible]

[illegible] items. [illegible]

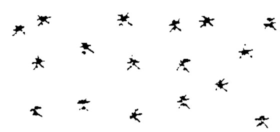


[illegible]

If necessary, continue the above activities of grouping objects and recording the numerals on the chalkboard.

The decimal numeration system, based on groupings of ten, is sometimes called the <u>base</u> <u>ten</u> system. We see that the base ten system uses only ten symbols. Can you name these symbols? (0, 1, 2, 3, [illegible], 5, 6, 7, 8, 9)

For additional practice the teacher may wish to duplicate a worksheet of examples similar to the one given below.

Group the following objects into sets of ten.	How many sets of ten are there?	How many single objects are there remaining?	How do you write the base ten numeral?
			

Suppose the shepherd used only one hand for counting purposes. Possibly, he would have worked with groups of five instead of groups of ten.

We may now compare this system with the base ten system. What name could we give it? (base five system) How many symbols did we use in the base ten system? (ten) How many symbols would we probably use in the base five system? (five) What symbols do you think we would use in the base five system? (0, 1, 2, 3, 4)

When the shepherd had counted out five sheep (one for each [illegible]), what would he do next? (He would place one rock in [illegible].) Why must we always have a symbol to represent [illegible]? ([illegible] is used to fill places which would otherwise be [illegible] and might lead to misunderstanding. For example, in [illegible], the zero is needed to show that although we have 2 hundreds and [illegible] ones we have no tens.)

Let us use our objects to illustrate the base five system. Put seven of them on your desk. Group the seven objects into sets of five. How many sets of five are there? (one) How many objects remain? (two) Add three more objects to this set. How many sets of five are there now? (two) How many objects remain? (none)

[illegible]

[illegible]

GROUPING IN BASE FIVE

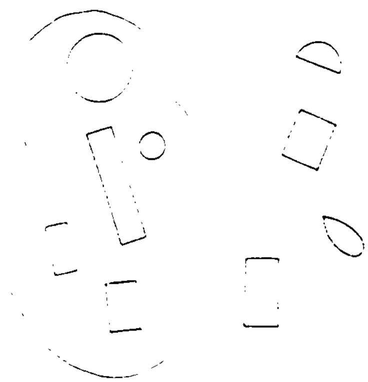
Our numeration system is sometimes called the <u>base ten</u> system. This means that we group in sets of ten. The <u>base ten</u> system uses only ten symbols. These are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. We can write any numeral by using the ones of these ten symbols that we need.

This picture shows how the base ten system is used in writing numerals. On your paper write the letters a, b, c, d, e, and f. Beside each letter write the numeral which belongs at that place in the table.

	Group these objects into sets of ten	How many sets of ten are there?	How many single objects are there remaining?	How do you write the base ten numeral?
1.	XXXXXXXXXX XXXXXX	1	6	16
2.	XXXXXXXXXX XXX XXXXXXXXXX	2	3	23
3.	XXXXXXXXXX XXXX	a ( )	b (4)	c (14)
4.	XXXXXXXXXX XXXXX XXXXXXXXXX	d (2)	e (5)	f (25)

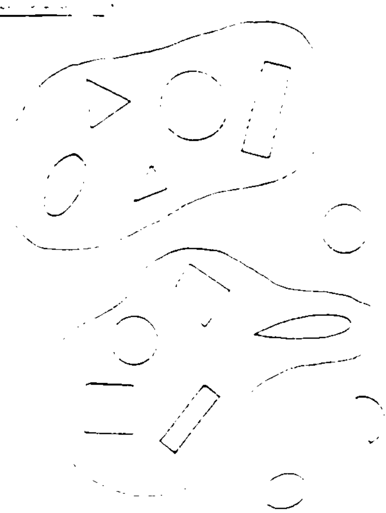
What would our numeration system look like if we tried to build a system which uses sets of five? Of course, in a base five system we would group by sets of five. Examples A, B, C, and D show how we can do this.

[illegible]



[illegible] sets of five. [illegible] the set [illegible].

[illegible]



In this picture, we [illegible] the sets of five objects [illegible].

[illegible]

[illegible]

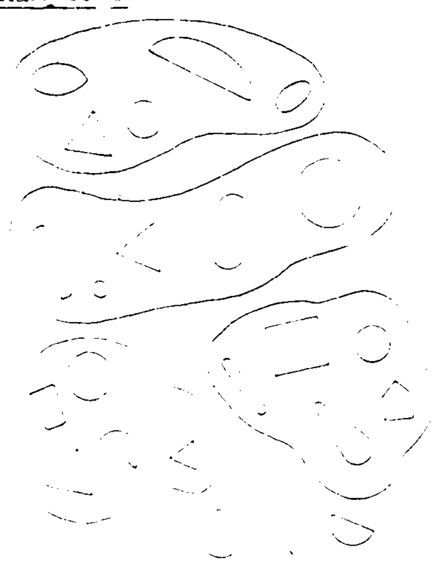
[illegible]

1. [illegible] (3)

2. [illegible] (2)

3. [illegible] (3) [illegible]
   (2) [illegible].

[illegible]



1. [illegible] (4)

2. [illegible] (3)

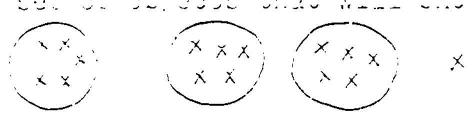
3. [illegible] (4) [illegible]
   (3) [illegible].

Exercise Set 1

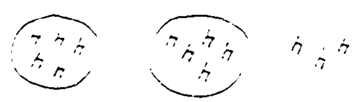
1. Copy and finish this table.

Group these objects into sets of five.	How many sets of five are there?	How many single objects are there remaining?
✓✓✓✓✓ ✓✓✓✓✓ ✓✓✓	2	3
OOOOO OOOO	(1)	(4)
xxxxx xxxxx xxxxx xx	(3)	(2)
◻◻◻◻◻ ◻◻◻◻◻ ◻◻ ◻◻◻◻◻ ◻◻◻◻◻	(4)	(2)

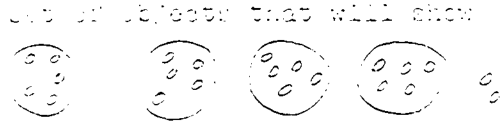
2. Draw a set of objects that will show 3 fives and 1 one.



3. Draw a set of objects that will show 2 fives and 3 ones.



4. Draw a set of objects that will show 4 fives and 2 ones.



5. Copy and complete this table.

Sets of Objects	Fives	Ones
XXXXX XXXX	1	4
OOOOO OOOOO OOOOO OO	3	2
✓✓✓✓✓ ✓✓✓✓✓ ✓✓✓✓✓ ✓✓✓✓✓ ✓	(4)	(1)
□□□□□ □□□□□ □□□□	(2)	(4)
△△△△△ △△△△△ △△△△△ △△△△△	(4)	(0)

It might be desirable to duplicate this exercise set for the pupils. In exercises 2-4 the pupils will enjoy designing their own objects to use in showing each set. Different objects may be used within the same set.

BASE FIVE NOTATION

Objective: To teach the children base five notation

Materials: Place-value chart and cardboard strips

Vocabulary: Notation

Now we are ready to introduce the writing of base five numerals. Use a place-value chart and bundles of cardboard strips to show groupings of fives and ones.

Fives	Ones

Write the base five notation on the chalkboard as the children count the number of strips. For example:

[illegible]	[illegible] five	[illegible] five	[illegible] five	10 five
[illegible]	[illegible]	[illegible] five	[illegible] five	[illegible] five
[illegible]	[illegible] five	22 five	[illegible] five	[illegible] five
[illegible]	[illegible] five	[illegible]	[illegible] five	[illegible] five
[illegible]	[illegible] five	[illegible] five	[illegible] five	

Limit the number of strips to twenty-four for this discussion. To count twenty-five strips using the base five notation, it would be necessary to know three-digit numerals which will not be introduced till grade five.

[illegible]:

Use a set of cardboard strips to illustrate the base five system. [illegible] begin with a single cardboard strip. [illegible] in the [illegible] place in the place-value box. This is [illegible] in base five. [illegible] The word [illegible] the numeral [illegible] of five. Add one strip. [illegible] another strip to the

[illegible] What symbol would we use? (3_{five}) Add another strip to the set. What symbol would we use? (4_{five}) Add another strip to the set. How many do we have? (One set of five) Put the 1 to represent the set of five in the fives' place. We have written 1_{five}, 2_{five}, 3_{five} and 4_{five}. How do we write one five and no ones? (10_{five}) We read this symbol as "one five and no ones," or "one zero, base five." What does 10_{five} mean? (one set of five and no ones)

> Continue adding a single strip each time and writing the name of the number represented using base five notation until $[illegible]_{five}$ is reached. Every time a set of five single strips are grouped, staple them and put them in the fives' place.

[illegible]

[illegible]	[illegible]	Numeral
I	1 one	1_{five}
II	2 ones	2_{five}
III	3 ones	3_{five}
IIII	4 ones	4_{five}
IIIII	1 five and 0 ones	10_{five}
IIIII I	1 five and 1 one	11_{five}
IIIII II	1 five and 2 ones	12_{five}
IIIII III	1 five and 3 ones	13_{five}
IIIII IIII	1 five and 4 ones	14_{five}
IIIII IIIII	2 fives and 0 ones	20_{five}

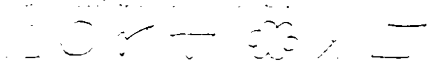
[illegible]

We need the symbols 0, 1, 2, 3, 4 in order to write numerals in the base five system.

[illegible] exercises has three parts. Write your [illegible]

[illegible]

[illegible]

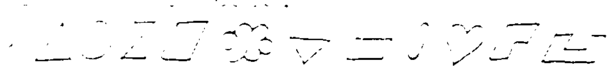


[illegible] sets of five are there? [illegible]

[illegible] remain? 2

[illegible] numeral written? Answer: [illegible] five

[illegible]

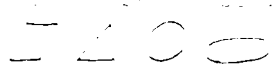


How many sets of five are there? (2)

[illegible] remain? [illegible]

[illegible] numeral written? (22 five)

[illegible]



How many sets of five are there? (2)

How many objects remain? (4)

How is the base five numeral written? (24 five)

[illegible] Think about eight cups.

a) How many sets of five are there? (1)

b) How many of the eight cups are not in the set of five? (3)

c) How is the base five numeral for eight written? (13 five)

[illegible] Think about nineteen cats.

a) How many sets of five are there? (3)

b) How many of the nineteen cats are not in a set of five? (4)

c) How is the base five numeral for nineteen written? (34 five)

	How many sets of five letters	[illegible]	[illegible]
			[illegible]
[illegible]	(2)	( )	(21 five)
[illegible]	(3)	(0)	(30 five)
	(3)	(4)	(34 five)
	(4)	(3)	(43 five)

The teacher should duplicate copies of this chart for pupils to complete.

(3 sixes and ones)

5. Finish these. In exercise a) you should think of how many sets of nine there are in twenty objects. Then think how many objects are remaining.

a) 20 = 22 nine   c) 20 = 26 seven

b) 20 = 24 eight   d) 20 = 32 six

Exercise Set [illegible]

[illegible] the base five numeral for each of these exercises.

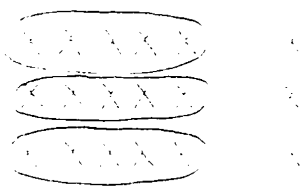
[illegible] = (4)$_{five}$	[illegible] 15 = (30)$_{five}$
[illegible] = (24)$_{five}$	[illegible] 7 = (3)$_{five}$
[illegible] = (32)$_{five}$	[illegible] 22 = (42)$_{five}$

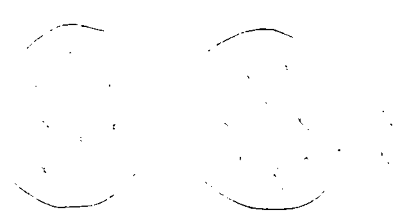
[illegible], in base ten, the number of objects which is meant by [illegible] of these base five numerals.

[illegible] 23_{five} = (13) objects	[illegible] $[illegible]_{five}$ = (6) objects
[illegible] $[illegible]_{five}$ = (8) objects	[illegible] 33_{five} = (18) objects
[illegible] $[illegible]_{five}$ = (2) objects	[illegible] $[illegible]_{five}$ = (21) objects

[illegible]

1. [illegible]

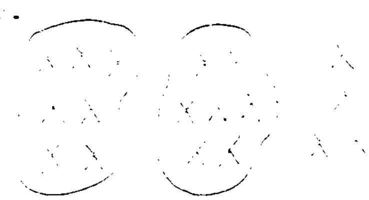




(2 eights and 2 ones = 22 eight)

(3 sixes and 0 ones = 30 six)

(2 nines and 0 ones = 20 nine)



(2 sevens and 4 ones = 24 seven)

2. Group [illegible] x's into the following sets. Each time find how many sets there are and how many remain. Then write the number.

a) [illegible] (40 eight) [illegible] (35 nine)

[illegible] (44 seven) [illegible] sets of six (52 six)

After you [illegible] Exercise Set I you will see that Exercise Set J is very much like Set I. Here we want [illegible] think of the [illegible] X's and [illegible] group them. [illegible] make the X's on your paper--just think [illegible].

[illegible] take [illegible] look to the last row of the table [illegible] Exercise Set J and see how we can [illegible] what numerals to write in the empty spaces.

Base Ten Numeral	Sevens	Ones	Base Seven Numeral
20			

The base ten numeral is 20. Think of 20 X's made on your paper--do not make them, just think of them. Next think of how many sets of seven there are in 20. Are there just 2 sets of seven each in 20? Yes. Also, there are 6 ones left over. How can you make a record of this in the table? You can put a 2 below Sevens and a 6 under Ones. Then below Base Seven Numeral you can write the numeral which means 2 sevens and 6 ones. This numeral is 26_{seven}. Then the next to the last row of the table will look like this:

Base Ten Numeral	Sevens	Ones	Base Seven Numeral
20	2	6	26_{seven}

Now complete the table in Exercise Set J.

[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	(2)	(^{12} three)
	[illegible]	[illegible]	(^{10} three)
	(2)	[illegible]	(^{22} three)
[illegible]	[illegible]	[illegible]	[illegible]
		[illegible]	(^{5} one)
	(3)	(4)	(^{34} [illegible])
	(3)	[illegible]	(^{30} one)
[illegible]	[illegible]		[illegible]
	(2)	[illegible]	(^{2} four)
	(3)	(2)	(^{32} four)
	(2)	(3)	(^{23} four)
[illegible]	[illegible]	[illegible]	[illegible]
	(3)	[illegible]	([illegible] eight)
	(2)	(0)	(^{20} eight)
	(2)	(3)	(^{23} eight)
[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	(6)	(^{16} seven)
	(2)	(6)	(^{2} seven)
	[illegible]	[illegible]	(^{13} seven)

A NUMBER MAY HAVE SEVERAL NAMES

Objective: To help children recognize that a number has several names

Materials: Place-value charts, bundles of sticks or cardboard strips, abacus

It is imperative for children to recognize that a number has several names. We might begin by bringing into the classroom seven objects, such as blocks or sticks, suitable for the entire group to see easily.

Throughout the discussion, such manipulative materials as place-value charts, an abacus, or bundles of sticks should be used to help children gain insight into the fact that there are many names for the same number.

To begin the discussion, ask the children to tell what can be written on the board to represent the number of objects that has been displayed.

Of course, there are many possible answers. Children may answer with the numeral 7, the word seven, 3 and 4, or perhaps VII. As children discuss different ways of naming 7, expressions such as these (8 - 1), (2 + 2 + 3), (7 x 1), etc., might be given as answers; but should not be encouraged by the teacher. These possibilities will be developed in later units. Our purpose here is primarily to emphasize grouping and writing in base ten and not computation. As suggestions are given, the children or teacher may record them on the chalkboard.

After the members of the class seem to understand various ways of expressing numbers whose base ten numerals have one digit, they may be led to explore ways of naming such "teen" numbers as sixteen. As different possibilities are suggested and recorded vertically on the board, a chart like the following may be developed.

Different Names for the Same Number

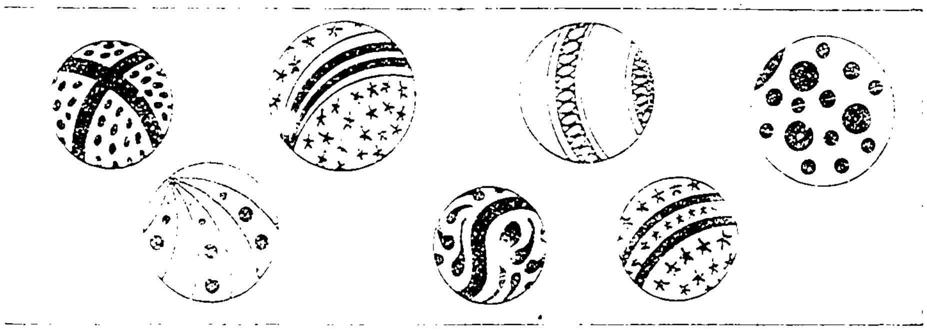
[illegible]

[illegible]

Use place-value boxes, bundles of sticks, or an abacus to explore ways of naming [illegible] three-digit decimal numeral. [illegible] should be placed on finding more [illegible] than [illegible] these numbers. The number [illegible] may be expressed as [illegible] hundreds, [illegible] tens, and three ones; [illegible] tens, and thirteen ones; [illegible] hundreds, [illegible] tens, and thirteen [illegible]. The latter [illegible] a review of [illegible] in subtraction.

[illegible] children seem to [illegible] familiar with [illegible] of expressing these numbers, [illegible] to find another name for [illegible] seven tens and eleven ones [illegible] is eight tens and [illegible]. Inclusion of such [illegible] as three hundreds, twelve tens, [illegible] ones, etc., is advisable, [illegible] meanings of each [illegible] renaming used in [illegible].

[illegible] is important that children develop [illegible] of these ideas. Exploring [illegible] materials will [illegible] exploration of [illegible] possibilities of different [illegible] for the same number should [illegible] individual [illegible].



Is each of these numerals a name for seven: 13_{four}, 21_{three}, 12_{five}, 11_{six}? Yes, they are. Some of these numerals are the kind of names you have been studying in this unit. They are all good names. Also these names are sometimes used for seven: VII and … .

There are still other names for seven. Is each one of these a name for seven: $8 - 1$, $3 + 4$, $10 - 3$, $5 + 2$? Yes. There are many more ways that we may name 7. What are some of the names? (Possible answers include these:

6 and 1
one seven
2 and 5
one less than eight)

Exercise Set 1

Write different names for the numbers. Use only base ten numerals. Example: [illegible]

1. Write __four__ different names for [illegible].

2. Write __four__ different names for 27.

3. Write __four__ different names for [illegible].

4. Write __four__ different names for [illegible].

5. Write __four__ different names for [illegible].

6. Write __four__ different names for [illegible].

7. Write __four__ different names for [illegible].

There are many correct answers for each exercise. Encourage variety of responses on the part of the pupils.

RENAMING NUMBERS

Objective: To help children understand the structure of the decimal system

Materials: Place-value box, cardboard strips, abacus

At this time we want the children to have further experiences in renaming. It is important that they understand that 225 can be renamed as

2 hundreds + 2 tens + 5 ones
22 tens + 5 ones
225 ones
1 hundreds + 12 tens + 5 ones
2 hundreds + 1 ten + 15 ones
200 + 20 + 5
100 + 120 + 5

Using a place-value box and bundles of sticks or cardboard strips will help to clarify this regrouping idea. The illustrative numbers used will need to be kept reasonably small, if cardboard strips or sticks are used. Provide experiences in a class situation having the children use these manipulative materials to show the various regroupings. Continue working with 4-place numbers to illustrate that this same type of renaming occurs with thousands. An abacus could be another teaching tool for developing these ideas.

Proceed from this concrete level to a more abstract level by recording on the chalkboard suggested renamings of numerals given by the children, as

2,345 = 2 thousands + 3 hundreds + 4 tens + 5 ones
= 23 hundreds + 4 tens + 5 ones
= 234 tens + 5 ones
= 2,345 ones
= 2000 + 300 + 40 + 5

RENAMING NUMBERS

<u>Example A</u>: The number 26 may be renamed as
2 tens and 6 ones.
1 ten and 16 ones.

<u>Example B</u>: Let us see how many ways we can think of for naming the number 457. In renaming use only ones, tens, and hundreds.

457 = 457 ones

457 = 4 hundreds + 5 tens + 7 ones

457 = 4 hundreds + 4 tens + 17 ones

457 = 3 hundreds + 15 tens + 7 ones

457 = 3 hundreds + 14 tens + 17 ones

457 = 45 tens + 7 ones

457 = 400 + 50 + 7

<u>Example C</u>: The number 4,605 can be renamed in many ways. Ones, tens, hundreds, and thousands can be used in renaming.

4,605 = 4,605 ones

4,605 = 46 hundreds + 0 tens + 5 ones

4,605 = 4 thousands + 6 hundreds + 5 ones

4,605 = 460 tens + 5 ones

4,605 = 4,000 + 600 + 5

4,605 = 4,600 + 5

Exercise Set 10

In exercises 1 through 10, write T if the sentence is true. Write F if the sentence is false.

(T) 1. 34 may be renamed as 3 tens + 4 ones.

(T) 2. 34 may be renamed as 2 tens + 14 ones.

(F) 3. 34 may be renamed as 4 tens + 3 ones.

(T) 4. 34 may be renamed as 1 ten + 24 ones.

(T) 5. 365 may be renamed as 3 hundreds + 5 tens + 15 ones.

(F) 6. 365 may be renamed as 35 tens + 5 ones.

(T) 7. 365 may be renamed as 2 hundreds + 15 tens + 15 ones.

(F) 8. 365 may be renamed as 3 hundreds + 60 tens + 5 ones

(F) 9. 184 may be renamed as 18 tens + 14 ones.

(T) 10. 184 may be renamed as 1 hundred + 7 tens + 14 ones.

For each exercise from 11 through 16 write the base ten numeral.

11. Six hundreds + five tens + three ones (653)
12. Three hundreds + twelve tens + eight ones (428)
13. Nine hundreds + four tens + fifteen ones (955)
14. 4 hundreds + 17 tens + 8 ones (578)
15. 7 hundreds + 5 tens + 16 ones (766)
16. 8 thousands + 6 hundreds + 3 tens + 12 ones (8,642)

Exercise Set 11

Name each of the following in two ways as a number of tens and a number of ones. Number 1 is answered for you.

1. 57 Answer:
   5 tens and 7 ones
   2 tens and 17 ones
2. [illegible] ([illegible] tens and 6 ones) ([illegible] tens and [illegible] ones)
3. [illegible] (5 tens & 4 ones) (4 tens & 14 ones)
4. [illegible] (8 tens & 0 ones) (7 tens & 10 ones)
5. 77 (7 tens and 7 ones) (6 tens and 17 ones)
6. [illegible] (3 tens and 9 ones) (2 tens and 19 ones)

Complete the following sentences.

7. [illegible] = [illegible] hundreds + (5) tens + [illegible] ones.

8. [illegible] = [illegible] hundreds + [illegible] tens + (14) ones.

9. [illegible] = [illegible] hundreds + (14) tens + 1[illegible] ones.

10. Name [illegible] in three different ways as hundreds, tens, and ones. (Answers will vary as in Ex. 7-9)

11. Name [illegible] in three different ways as hundreds, tens, and ones. (Answers will vary as in Ex. 7-9)

12. Name the number [illegible] in four different ways. (Answers will vary.)

13. Name [illegible], [illegible] in three different ways as thousands, hundreds, tens, and ones. (Answers will vary.)

14. Name the number [illegible],[illegible] in four different ways. (Answers will vary)

15. Name each of the following as a base [illegible] numeral.

a) Four [illegible] + [illegible] + six ones = (456)

b) Two thousands + three hundreds + seven tens + five ones = (2,375)

c) [illegible] hundreds + [illegible] tens + [illegible] ones = (733)

d) Nine hundreds + [illegible] + fourteen ones = (1,044)

e) [illegible] + [illegible] + [illegible] = (1,359)

Exercise Set 12

1. From the list below write all the letters which are beside correct names for 467.

    a) Four hundred sixty-seven

    b) Forty-six and seven more

    c) Forty-six tens and seven

    d) Forty hundreds + sixty-seven ones

    e) 300 + 160 + 7

    f) Seven plus four hundred

    g) 400 + 60 + 7

    h) 300 + 150 + 17

    i) 467 tens

2. Answer Yes or No.

    a) 3,729 is 37 tens plus 29 ones. (no)

    b) Ten hundreds plus forty tens plus nine ones is the same as one thousand forty-nine. (no)

    c) 5,000 + 500 + 1 = 5,501 (yes)

    d) 35 hundreds + 1 ten + 18 ones = 3,528 (yes)

    e) 734 = 600 + 120 + 24 (no)

3. Write the base ten numeral for each.

    (5,683) a) Five thousands + six hundreds + eight tens + three ones

    (3,965) b) 3 thousands + 8 hundreds + 16 tens + 5 ones

    (7,527) c) 6 thousands + 15 hundreds + 2 tens + 7 ones

    (8,556) d) 8 thousands + 4 hundreds + 14 tens + 16 ones

    (10,244) e) 9 thousands + 12 hundreds + 3 tens + 14 ones

EXTENDING IDEAS OF THE DECIMAL SYSTEM

Objective: To develop an understanding of place-value and a proficiency in reading and writing large numbers

Materials: A chart showing place and group names may be helpful.

Vocabulary: digit

> If this word is not understood by the class, explain that a digit is any one of the ten symbols we use to write a numeral.

Since the decimal system is used in most parts of the world, we need to understand it. So far, we have talked about numbers whose numerals have one, two, three, and four digits. We know that the position of a digit in a numeral determines its place-value. We know also that ten ones is the same as one ten and that ten tens is the same as one hundred. In like manner, ten hundreds is the same as one one-thousand, ten one-thousands is the same as one ten-thousand, and ten ten-thousands is the same as one hundred-thousand.

1 hundred-thousand	1 ten-thousand	1 one-thousand	,	1 hundred	1 ten	1 one

The numerals in the decimal system may be read more easily if they are grouped in sets of three digits beginning at the right. Each set of three digits is separated from the rest of the numeral by a comma or commas.

To read a six place numeral, we start with the group on the left, reading the first set of digits as one numeral, followed by the word "thousand" as "one hundred eleven thousand." Then we read the second group of digits as one numeral, "one hundred eleven." The complete numeral is read "one hundred eleven thousand, one hundred eleven."

Some examples which may be used for purposes of discussion should include such numerals as:

162,142	162,000 + 142 read one hundred sixty-two thousand, one hundred forty-two
234,172	234,000 + 172 read two hundred thirty-four thousand, one hundred seventy-two
23,100	23,000 + 100 read twenty-three thousand, one hundred
2,020	2000 + 20 read two thousand, twenty
400,001	400,000 + 1 read four hundred thousand, one

It may be necessary to give more explanation and to devote more time to reading five- and six-place numerals

EXTENDING IDEAS OF THE DECIMAL SYSTEM

We know that each place in a numeral written in the decimal system has a name. In a whole number the first place on the right is the ones' place, the second the tens' place, and the third the hundreds' place. The fourth position is the thousands' place, the fifth the ten thousands' place, and the sixth is the hundred thousands' place.

We also know that:

10 <u>ones</u> are the same as 1 <u>ten</u>;

10 <u>tens</u> are the same as 1 <u>hundred</u>;

10 <u>hundreds</u> are the same as 1 <u>thousand</u>;

10 <u>thousands</u> are the same as 1 <u>ten</u> <u>thousand</u>;

10 <u>ten</u> <u>thousands</u> are the same as 1 <u>hundred</u> <u>thousand</u>.

[illegible]	[illegible]	[illegible]	[illegible]	Hundreds	Ten	One
[illegible]	[illegible]	[illegible]	[illegible],	2	2	2

[illegible], each 2 has a different [illegible]. Tell [illegible].

We now want to read the names of large numbers. These will be numerals with as many as six digits but with no more than six.

You can read large numerals of as many as six digits easily if the three digits at the right are separated from the others by a comma, as in 222,222 or in 74,609. The number named by the digits to [illegible] left of the comma tell how many <u>thousands</u> there are; the number named by the digits to the right of the comma tells how many [illegible] there are.

We read 222,222 as "two hundred twenty-two <u>thousand</u>, two-hundred twenty-two." We read 74,609 as "seventy-four <u>thousand</u>, six hundred nine." Notice that the word "and" is <u>not</u> used in any way in reading numerals like these.

Read these numerals.

736,421 (seven hundred thirty-six thousand, four hundred twenty-one)

80,592 (eighty thousand, five hundred ninety-two)

603,250 (six hundred three thousand, two hundred fifty)

248,759 (two hundred forty-eight thousand, seven hundred fifty-nine)

10,003 (te~ thousand, three)

5,791 (five thousan~, seven hundred ninety-one)

890,602 (eight hundred ninety thousand, six hundred two)

927,030 (nine hundred twenty-seven thousand, thirty)

10

Exercise Set 17

Write the letter which is in front of the correct way to say each of these numerals. Exercise 1 is done for you.

1. 28
   a) twenty-seven   Answer: 1. c
   b) eighty-two
   c) twenty-eight
   d) eight

2. 5,250
   a) five thousand, two hundred five
   b) five hundred twenty-five
   c) five thousand, one hundred fifty
   d) five thousand, two hundred fifty

3. 17,002
   a) seventeen thousand, two hundred
   b) seventeen thousand, two
   c) seventeen thousand, twenty
   d) one hundred seventy thousand, two

4. 156,946
   a) one hundred fifty-six thousand, nine hundred forty-six
   b) one hundred sixty-five thousand, nine hundred forty-six
   c) one hundred fifty-six million, nine hundred forty-six
   d) one fifty-six thousand, six hundred forty-nine

5\. [illegible] [illegible]

thousand

c) [illegible]

d) [illegible] thousand, [illegible]

d) [illegible] million, [illegible]

6\. [illegible] a) two hundred thousand, seven hundred

b) two hundred thousand, seventy

c) two hundred thousand, seven

d) one thousand, seven

7\. [illegible] a) three hundred fifty-six thousand, eight hundred forty-three

b) three hundred sixty-five thousand, eight hundred forty-three

c) three hundred sixty-five thousand, eight hundred forty-four

d) three hundred sixty-five million, eight hundred forty-three

8\. [illegible] a) twenty-two thousand, two hundred twenty-two

b) two hundred twenty-two thousand, twenty-two

c) two hundred twenty-two million, two

d) two hundred two thousand, two

Write numerals to represent each of the following.

9. Three hundred six (306)

10. Six thousand, seven hundred fifty-six (6,756)

11. Forty-seven thousand, six hundred four (47,604)

12. Forty thousand, one hundred twenty-five (40,125)

13. Three hundred fifty-one thousand, five hundred sixty-four (351,564)

14. Forty thousand, forty (40,040)

15. Two hundred fifty thousand, fifty-six (250,056)

16. Seven hundred fifty-six thousand, thirty (756,030)

17. Five hundred thousand, five (500,005)

18. [illegible] ([illegible]04,004)

19. Six thousand, six (6,006)

20. Sixty thousand, six (60,006)

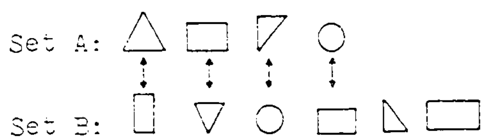
21. Sixty thousand, six hundred six (60,606)

ORDER RELATIONS ON THE NUMBER LINE

Objectives: To clarify order relations with the use of a number line

Materials: Sets of checkers and checker board; flannel board with cut-outs; a number line

Until now, grouping in terms of bases has been emphasized. In grouping, children compare numbers by matching the elements in one set with those in another set on a one-to-one basis. The set with more elements is the larger one; or, the set with fewer elements is the smaller one.



There are four discrete objects in Set A and six discrete objects in Set B. (Discrete means distinct or separate.) Children need experiences comparing sets of discrete objects to determine which set is larger or smaller.

Another way to think about numbers is to associate them with points on a line. Each number is associated with a unique (i.e., one and only one) point on a "number line," and vice versa. Of two unequal numbers one is the greater. We can add a number to the smaller to get the greater. The greater is to the right of the smaller.

In many cases, the number line will have been introduced in the work of the lower grades. It is important that the child grasp clearly the idea of the number line, as it will be used repeatedly as a graphic aid in:

(a) comparing numbers and numeration systems (Chapter 2)

(b) clarifying addition and subtraction (Chapters 3 and 6) and multiplication and division (Chapters 4 and 7)

You have already learned about sets of objects, like marbles, checkers, and so forth. Suppose we give John a handful of red checkers and Sally a handful of black checkers. Are there more black checkers, or more red checkers, or the same number of each? How do you find out? (Count the number of black checkers, and then the number of red checkers, and see which, if either, is larger.)

Let's pretend for a moment that we don't know about the one, two, three, and so on, used in counting. Then we cannot count the checkers. How could you find out whether there are more black checkers or red checkers? (You could do this by <u>matching</u>.) First, you could match a black checker with a red checker by putting them side by side, then another black checker with another red checker, and so on. What would happen at the end of this matching? (There would be either more black checkers or more red checkers or they would come out even.) What would this tell you? (It would tell whether there are more black checkers, or more red checkers, or the same number of each.)

> If necessary, you can easily devise some class demonstrations or exercises in actual matching, using sets of stars, circles, etc., on a flannel board. A classroom activity might involve matching pupils and chairs in a one-to-one way.

Even without numbers, we have a way of seeing that some sets have more members than others. Of two such sets, we say that the one with more members is the one of <u>larger</u> <u>size</u>, and that the one with fewer members is the one of <u>smaller</u> <u>size</u>. We see that sets can be of various sizes.

> "Size", in connection with more or fewer members, should not be confused with the physical size of the objects in the sets: a set of six mice is of "larger size" than a set of two elephants!

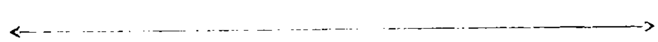
What is the smallest-sized set you can think of which is not empty? (a set with just one member) What is the next smallest-sized set? (a set with just two members) What is the next after that? (a set with just three members) The numbers one, two, three, and so on, are really just the various possible sizes of sets, arranged in order beginning with the smallest.

Now let us think how we count a set of objects. What do we do first? (We first match the numeral one with an object in the set.) Then, if there are any objects left, what do we do next? (We match the numeral two with another object in the set.) If there are still any objects left, what do we do next? (We match the numeral three with still another object in the set, and so on.) The last numeral matched with an object in the set is then the size of the set. It tells how many objects there are in the set.

This shows that counting is matching, too. It is matching a set of number names with objects. The numerals one, two, three, and so on, used in counting, represent counting numbers.

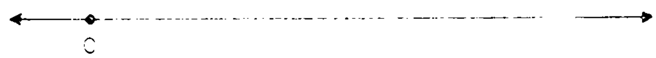
What are the numbers we use in counting? We all know that they are the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, and so on. These numbers together with the number zero form a set which we call the set of whole numbers.

In developing a number line, we can use a straight line with points corresponding to whole numbers marked along it. To do this we first draw a picture of a line.

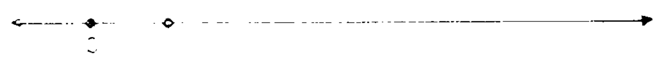


Draw a picture of a line on the chalk-board as shown above. (Of course, we can really represent only a piece of a line, but we put arrows at the ends of the piece to show that the line "keeps on going" beyond these ends to the right and to the left.) As the discussion proceeds, the successive drawings below will indicate how you will add the labeled points to this number line.

Next we choose a point on the number line to represent the number zero. We call this the point labeled zero, or just the point zero.

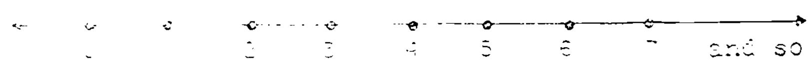


Then we choose a point a short distance to the right of this to represent the number one. We call this the point one.



The same distance to the right of the point labeled 1 we mark another point and this will be the point two.

Continuing in this way, we get what we call a number line.



and so on.

On the chalkboard we have a number line showing all the numbers from 0 through 100.

> Number lines may be obtained from several sources. Alternatively, such a number line may be drawn on the chalkboard or constructed from available materials by the teacher. You may want to use a thermometer as an example of a vertical number line.

Leading Questions:

1. What is the smallest whole number represented on the number line? (zero)

2. In representing a number line, do we *have* to show the point labeled zero? (No. It is possible to show just any convenient part of the number line. If we start with the smallest whole number, then we start with zero. But if we label the first point 202, then the next points to the right, in order, will be 203, 204, 205, and so on.)

170

3. What do we know about the numbers represented by the points to the right of the first labeled point on the number line? (They are larger.)

4. Which of the numbers 6 and 9 is the greater? (9. We must add 3 to 6 to get 9.) The point 9 is to the right of 6 on the number line. If one number is greater than another, what do you know about their places on the number line?

5. What does the arrow on the right side of the number line mean? (The arrow indicates that we have drawn only a piece of the number line, and that the line, with its labeled points, continues to the right.)

6. Is there a largest number on the number line? (No. For every whole number represented on the number line there is also a next larger whole number.)

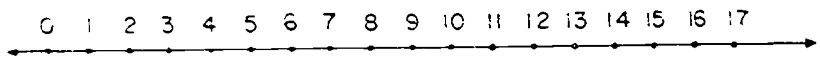
7. What do you think the arrow on the left side of the number line tells us? (It tells us that the number line keeps on going in that direction.) Later, we will have use for the part of the number line extending to the left of zero.

8. What can you tell me about <u>every</u> number represented by a point that lies to the right of a given point? (It is larger than the number represented by the given point.)

9. What can you tell me about <u>every</u> number represented by a point that lies to the left of a given point? (It is smaller than the number represented by the given point.)

We can use the number line to see which of two numbers is the larger and which the smaller. On the number line, <u>the point standing for the larger number always lies to the right of the point standing for the smaller number</u>. We say that the smaller number is <u>less than</u> the larger and that the larger number is <u>greater than</u> the smaller.

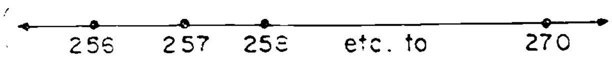
ORDER RELATIONS ON THE NUMBER LINE



We have talked about the number line. We know that it is a straight line with points on it which have been matched with the set of whole numbers. The point zero is at the left. If you think of two numbers represented on a number line, the one to the right is larger than the one to the left.

In drawing a number line, do we <u>have</u> to show the point that is labeled zero? Of course we do not. It is possible to show just a part of the number line. If we start with the smallest whole number, then we start with zero. But if we label the first point 132, then the next points to the right, in order, will be 133, 134, 135, and so on.

<u>Exercise Set 14</u>

1. Draw a number line showing the whole numbers from 256 through 270. ()

In each of the following exercises, find the points representing the two numbers on the number line. Write the missing numbers on your paper.

2. 266 and 258

   <u>(258)</u> is less than <u>(266)</u>

   <u>(258)</u> is to the left of <u>(266)</u>

   <u>(266)</u> is to the right of <u>(258)</u>

3. 261 and 269

(261) is less than (269)

(269) is to the right of (261)

(261) is to the left of (269)

4. 270 and 263

(270) is greater than (263)

(263) is to the left of (270)

(270) is to the right of (263)

5. 264 and 268

(264) is less than (268)

(264) is to the left of (268)

(268) is to the right of (264)

6. 257 and 260

(260) is greater than (257)

(260) is to the right of (257)

(257) is to the left of (260)

SOME NEW SYMBOLS

We usually write "2 + 5 equals 7" using the equals sign (=) instead of writing out the word "equals". We write:

$$2 + 5 = 7$$

We have another symbol which is called the "less than" sign. To write "2 is less than 3," we write

$$2 < 3.$$

We also have a "greater than" sign. To write "8 is greater than 3," we write

$$8 > 3.$$

Complete the following statements with "less than" or "greater than".

1. $5 < 7$ Five is (less than) seven.
2. $12 > 9$ Twelve is (greater than) nine.
3. $0 < 1$ Zero is (less than) one.
4. $6 > 0$ Six is (greater than) zero.
5. $201 > 198$ Two hundred one is (greater than) one hundred ninety-eight.
6. $5 < 327$ Five is (less than) three hundred twenty-seven.

Exercise Set 15

Write each of these sentences in the short way.

EXAMPLE: Seven is less than ten. $7 < 10$

1. One is greater than zero. $(1 > 0)$

2. Eleven is less than thirteen. $(11 < 13)$

3. Fifty-six is greater than twenty-one. $(56 > 21)$

4. Two hundred sixty is less than three hundred sixty. $(260 < 360)$

5. Three hundred fifty-nine is greater than two hundred ninety-seven. $(359 > 297)$

6. Two hundred sixty-two is less than three hundred. $(262 < 300)$

Write the numerals 1 to 12 in a column on your paper. Write T if the statement is true and F if the statement is false.

1. $4 < 7$ (T)
2. $5 > 1$ (T)
3. $4 > 4$ (F)
4. $0 < 5$ (T)
5. [illegible]0 < 27 (F)
6. $52 > 49$ (T)
7. 12[illegible] < 19 (F)
8. 299 > [illegible]17 (F)
9. [illegible] > [illegible] (T)
10. [illegible] < [illegible] (F)
11. [illegible] > [illegible] (T)
12. [illegible] < [illegible] (T)

Exercise Set 16

Copy each mathematical sentence below. Fill in the "less than", "greater than", or "equals" sign in order to make a true statement.

1. (8 + 4) __(>)__ 11

2. (1 + 3) __(<)__ (9 + 1)

3. (2 × 3) __(<)__ (3 × 3)

4. (2 + 7) __(=)__ (11 − [illegible]

5. (2 + [illegible]) + [illegible] __(=)__ 1 + ([illegible] + [illegible])

6. (3 × [illegible]) __(>)__ (2 × 5)

7. (12 + 50) __(>)__ 59

8. (20 + 15) __(=)__ (10 + 25)

9. (8 + 5) − 2 __(<)__ (7 + 5) + 2

10. (24 − 2) __(>)__ ([illegible] × 7)

11. (18 − [illegible]) __(=)__ (9 + [illegible])

12. ([illegible] + 8) − 5 __(<)__ (14 − 7) + 5

JUST FOR FUN

Make a copy of this puzzle and fill it in.

A 2	B 7	3			C 4	D 3	E 9
F 4	9		G 2	H 6		I 6	9
0		J 4	6	1	K 7		9
	L 2	1			M 2	N 7	
	O 3	0			P 8	2	
Q 1		R 3	S 5	T 6	9		U 2
V 5	2		W 1	2		X 1	7

Graph paper could be used to make it easier for the children to copy the puzzle or the teacher could duplicate the puzzle.

Across

A. Two hundred sev[illegible]-three
C. Another name for [illegible]
F. [illegible] tens + [illegible] ones
G. [illegible] eights and [illegible]
I. [illegible] = [illegible]
J. [illegible] hundreds + [illegible]
L. ___ is [illegible] groups [illegible]
M. [illegible] nickels and [illegible]
O. ([illegible] × 7) + [illegible]
P. ___ is 11 more than [illegible]
R. [illegible]
V. [illegible] = ___ tens + [illegible]
W. How many [illegible]
X. [illegible] is another [illegible] ___

Down

A. [illegible] tens
B. ___ < [illegible]
D. [illegible] dozen
E. [illegible] less than one thousand
G. 2 tens and [illegible] ones
H. ___ > [illegible]
J. Four thousand, one hundred three
K. [illegible],289; 5,289; [illegible],289; ___
L. [illegible] ten and 13 ones
N. [illegible] equals how many tens?
[illegible]. [illegible] fours + [illegible] ones
[illegible]. [illegible] = ___ hundreds + [illegible] tens + [illegible] ones
T. Five tens and twelve ones
U. [illegible], 12, 17, 22, ___

Chapter 3

PROPERTIES AND TECHNIQUES OF ADDITION AND SUBTRACTION

PURPOSE OF THE UNIT

The purpose of this unit is to help children understand the nature of addition and subtraction as operations of mathematics. In so doing, they will also become acquainted with the fundamental properties of these two operations and their relationship to each other.

No doubt children have had experience with these operations but they need to consider the nature of addition and subtraction carefully to expand and reinforce their concepts of addition and subtraction. It is assumed that this is the first time emphasis will have been given to these properties of addition: closure property, commutative property, associative property, and the property of zero in addition.

The developmental material of the chapter uses the following procedure in thinking about a problem situation: state the number relationship in the problem by a mathematical sentence; decide what operation to use; perform this operation on the numbers; and, use the number which results to express the idea in the original problem situation. In general, but with individual adaptations, children will follow the preceding procedure. There are two objectives to be reached with the children: (1) the analysis of the problem and the expression of the relationships in the problem in a mathematical sentence, and (2) the solution of the problems by the use of the mathematical sentence. The person who has developed skill in solving problems not only has many plans but frequently uses his steps in a different order in solving different problems. The plan, as suggested, is a means for helping pupils extract numbers from the action implied in the problem and record their thinking in a systematic manner.

MATHEMATICAL BACKGROUND

The emphasis in this unit is on the operation of addition and the relationship of addition and subtraction. It might be re-emphasized here that the concept of addition may be thought of by considering the cardinal number of elements in each of two disjoint sets and that the sum of these numbers is the cardinal number of elements in the union set of the two disjoint sets. (Two or more sets that have no common elements are disjoint sets, i.e., there is no element in one set that is also in the other set.) The operation of addition is always on <u>just two</u> numbers and consequently is called a <u>binary operation</u>. We must of course, have a set of elements on which addition is defined and in this unit the set is the set of whole numbers $W = \{0, 1, 2, 3, \ldots\}$. "A binary operation on a set is a rule whereby to each ordered pair of elements of the set there corresponds exactly one element." (SMSG Studies, Vol. III, p. 17). In the possible cases we operate upon two numbers to produce a unique third number. In subtraction and division clearly the order is important. For although $12 + 9 = 9 + 12$ it is not true that $12 - 9 = 9 - 12$ if our set is W, since $9 - 12$ is not a number in W. The use of the phrase <u>binary operation</u> with pupils at this stage should be left to the teacher's judgment but recognition by the pupils that the addition operation is defined on <u>just two</u> numbers to produce a third number needs to be well established. It is this recognition that emphasizes the need for the associative property for addition. For how do we add 3, 4, and 2 if addition is a binary operation? The answer to the question depends upon grouping, or associating, <u>just two</u> of the numbers to get one number and then pairing this number with the third number: $3 + 4 + 2 = (3 + 4) + 2 = 3 + (4 + 2)$. The same remarks will apply to the operation of multiplication to be considered later.

When studying operations with children, we shall describe the operations as ways of thinking about two numbers to get another number. When the child adds 4 and 5 to get 9 or subtracts 9 and 5 to get 4 he simply thinks of 9 and 5 in two different ways. Addition is an operation on two known addends (4 and 5) to produce a third number called the sum. Subtraction is an operation for finding an unknown addend if the sum and the other addend are known (9 sum, other addend 5, unknown addend 4). If this is understood the need to emphasize the so-called subtraction facts is eliminated. For example, 13 - 6 may be thought of as "What number (unknown addend) added to 6 (known addend) is 13 (sum)?" Hence, if one knows 7 + 6 = 13, he also knows 13 - 6 = 7 and 13 - 7 = 6.

267	addend
+394	addend
661	sum
661	sum
-267	addend
394	addend

The inverse relationship of addition and subtraction is important to an understanding of these operations. Although the term <u>inverse</u> is not used with children, they can learn that subtraction of a given number will undo addition of that same number, and addition of a number will undo subtraction of the same number; e.g., if 5 is added to 3, the addition may be undone by subtracting 5. (3 + 5) - 5 = 3. Or, if 4 is subtracted from 9, the subtraction may be undone by adding 4. (9 - 4) + 4 = 9.

The property of <u>closure</u> explains that there is no limitation on the operation of addition in the set of whole numbers but there is a limitation on the operation of subtraction when only the set of whole numbers is used. Any two whole numbers can be added to produce another whole number; but we cannot always subtract two whole numbers and obtain another whole number; e.g., if we subtract 5 from 2 the result is not a whole number. Mathematicians express this fact by saying the set of whole numbers is not closed under

[illegible]. Later in his study of [illegible] the child will be introduced to [illegible] and [illegible] the particular sets will be closed under subtraction and division. However, all we expect the children to understand now is that in using only the set of whole numbers, the operation of addition in the set of whole numbers is always possible but the operation of subtraction is not always possible in the set of whole numbers.

The commutative property for addition states that the order of addends in a two-addend sum may be changed without a change in the sum: e.g., $7 + 2 = 2 + 7$ or, in general, $a + b = b + a$. It should be obvious that there is no commutative property for subtraction; e.g., $3 - 2 \neq 2 - 3$. There are two advantages of the commutative property for addition: (1) the number of addition facts the child needs to learn is fewer and (2) in some additions such as $2 + [illegible]$ the addition might be performed more easily if the child thinks of $[illegible] + 2$.

It has been emphasized that addition is an operation upon two numbers, producing a third number. What if we want to add three numbers? We must do the operation on two numbers at a time; i.e., we group the numbers appropriately. For example, using 3, 4, and 7 (keeping the numbers in that order) we might have $(3 + 4) + 7$ or $3 + (4 + 7)$. Will these two different groupings produce the same result? Yes. In fact, this is exactly what the associative property for addition states, namely, $(a + b) + c = a + (b + c)$. Since without changing the order, the only two possible groupings of three numbers give the same result, it is convenient and customary to write $3 + 4 + 7$ when either $(3 + 4) + 7$ or $3 + (4 + 7)$ is really meant. Incidentally, similar remarks apply for the sum of four or more numbers. It is also true that there is no associative property for subtraction: e.g., $(5 - 3) - 2 = 0$ whereas $5 - (3 - 2) = [illegible]$. (Note that the expressions in parentheses represent one number.)

The use of zero as an addend in addition or subtraction is important. It is the only addend which, when used, results in no change to one addend in addition ($[illegible] + 0 = [illegible]$) or to the sum in subtraction ($6 - 0 = 6$).

A sentence about numbers is a mathematical sentence. In the box are some examples of mathematical sentences.

$3 + 2 = [illegible]$
$14 - 6 = [illegible]$
$2 + n = [illegible]$
$[illegible] \div 12 = [illegible]$
$[illegible] + [illegible] = [illegible]$
$3 > 1 + n$
$[illegible] \neq 6 + 1$

In some mathematical sentences in this chapter which expresses operations or relationships, an n or some other letter of the alphabet will be used as a symbol for any member of the set of whole numbers.

Some mathematical sentences are true. $[illegible] + [illegible] = [illegible]$, $15 - 7 = [illegible]$, $[illegible] \times 2 = 16$, $6 > 2$, or $[illegible] = 2 + 1$ are all examples of true mathematical sentences.

Some mathematical sentences are false. $[illegible] + 7 = 6$, $12 - [illegible] = 1$, $6 \times 1 = 7$, $12 - [illegible] = 0$, $6 < 6$ are all examples of false mathematical sentences.

Some mathematical sentences are called open. They are open because you do not know if they are true or false. Some examples are: $[illegible] - n = 1$, $2 + [illegible] = n$, $n - [illegible] = [illegible]$. If $n = 1[illegible]$ then $[illegible] + [illegible] = n$ is a true sentence. If $n = 6$, then $[illegible] - n = 1$ is a false sentence.

In this chapter you will be interested in answering questions such as, "If $n = 6$, is $n - 2 = [illegible]$?", or "If $2 + 7 = n$, is $n = 9$?", or "What number must n represent so that $n - 7 = [illegible]$?"

The terms false and open mathematical sentences will not be used. The term true mathematical sentence will be used. Unless otherwise stated, all mathematical sentences in this chapter are true, because in all open sentences such as $n + 2 = 7$ it is intended that the replacement be such a number that the sentence will be true.

TEACHING THE UNIT

This is the first of two units in Grade - which extends the ideas of addition and subtraction of whole numbers and makes more explicit the properties and techniques of these operations. The suggestions on the following pages are designed to help you in teaching the unit.

Although the general plan of the commentary is similar to those for other units, we do wish to make a special note regarding some of the developmental materials. The exploration for some sections has been developed through a series of exercises for class discussion. Should a teacher prefer to develop these ideas independently of the exercises, then the exercises in the developmental sections may be used for children to do "on their own."

We suggest that a teacher study each section carefully and decide in what way the most effective development can be planned for his particular class. It certainly is intended that a teacher will use his own ingenuity and creative ideas to the fullest--without making significant changes in the intent of the unit.

ADDITION AND SUBTRACTION

Objective:

(1) To help children see that addition can be used to find the number of [illegible] in the union of two sets in which no [illegible] is a member of both sets.

(2) To help children see that addition of the numbers of elements in two sets does not give the number of elements in their union if the sets have members in common.

(3) To help children use addition by use of the number line 2.

Vocabulary: Union of sets, intersection of sets, common members

Teaching Procedures:

The teacher should provide sets of objects or drawings of sets of objects on the board. The sets should have no element in common. Each set has a cardinal number, the number of elements in the set. The union of the two sets has a cardinal number, the sum of the cardinal numbers of the two sets.

The children should review the ideas of Chapter I on union of sets and intersection of sets.

Exploration:

If we form a union of a set of [illegible] science books and a set of [illegible] spelling books, the result is a set of [illegible] books. [illegible] is obtained by adding [illegible] and [illegible]. We are not adding sets; we are adding numbers. The addition of numbers, however, helps us to describe the union of some sets.

Let us look at another example of the union of sets.

	Number of Members in Each Set	Number of Members in Union of Sets
Tom has 30 new toy soldiers and John has 25 old toy soldiers. How large a toy army can they form?	30, 25	55

In [illegible] a set [illegible] members was formed by the union of two sets, one of [illegible] members and [illegible] of [illegible] members. Do we use addition in this example? (Yes. [illegible] and [illegible].) What [illegible] result [illegible]

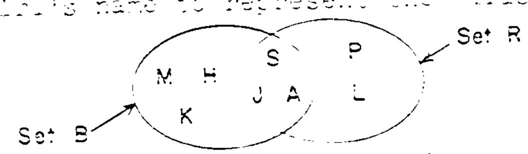
In [illegible] examples we are using addition to find the number of members of the union of two sets.

I would like to ask you a question about using addition in this way. Can addition [illegible] used to find the number of members in the union of any two sets.

> The children should give examples recalling their work on the union of sets. If they do not offer an example which involves sets having some members which are the same, the teacher should offer an example similar to the one following.

I have an example of the union of two sets which we should study. Here it is. I am thinking of a set of girls (B) whose members have blue eyes and a set of girls (R) whose members are wearing red dresses. Set B = {Mary, Jane, Sue, Ann, Helen, Kay}. Set R = {Paula, Sue, Jane, Ann, Linda}. Show the union of Set B and Set R. (The union of Set B and Set R = {Mary, Jane, Sue, Ann, Helen, Kay, Paula, Linda}.)

> The teacher should draw a picture to represent the union of these sets. Below is one way it may be done using the first letter of each girl's name to represent the girls.



> It is noted Sue, Jane, and Ann are pictured as members of both sets. They are members of the union of the two sets. They are included only once in the union of the two sets.

Let us go back to our question. Can we use addition to find the number of members in the union of any two sets? How many members are in Set B? (6) How many members are in Set R? (5) How many members are in the union of Set B and Set R? (8) Do we add 6 and 5 to get the number of members in the union? (No) Why? (Because they have some members which are the same i.e., they have some <u>common</u> <u>members</u>)

Let us think of other examples of the union of sets for which addition is not used to find the number of members of the union.

What is the answer to my question. Can we use addition to find the number of members in the union of two sets. (Not always. We can with some. We can't with others.)

How can you finish the following statement? "Addition may be used to find the number of members in the union of two sets which ... (have no members the same or have no common members.")

The union of sets gives us an opportunity to use subtraction. For example, if the union of a set of [illegible] cups and a set of n plates is a set of 12 objects, then the unknown set must have 8 plates. Eight is obtained by using the operation of subtraction. We are not subtracting objects or sets; we are subtracting numbers.

> The teacher may wish to examine examples of the union of sets in a manner similar to that used in the development of the use of addition in situations indicating the union of sets.

Can we use addition to find the number of members in the union of just <u>any</u> two sets? (No, but we can for any two sets with no members the same.) Does this limit apply when subtraction is used to find the number of members of an unknown set of the union of two sets? (Yes. The two sets of the union must have no members the same.)

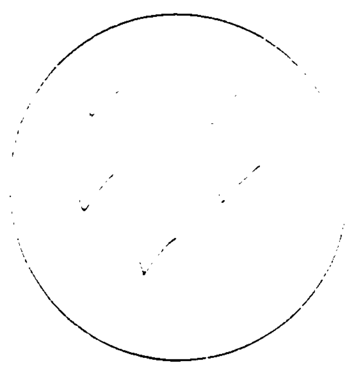
> Exercise Set I and II now be done by pupils.

Chapter 8

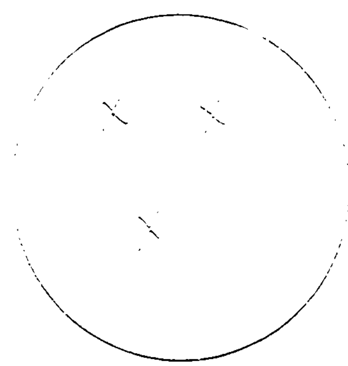
PROPERTIES AND TECHNIQUES OF ADDITION AND SUBTRACTION I

ADDITION AND SUBTRACTION

1.



Set A



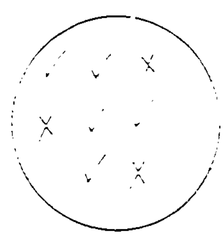
Set B

(a) How many members are in Set A? (5)

(b) How many members are in Set B? (3)

(c) How many members are in the union of Set A and Set B? (8)

A picture of $A \cup B$ is:



$A \cup B$

When we find the number of members in the union of two sets which have no common members we may use addition.

2. Set G is a set of nine year old girls in a fourth grade room.

   G = {Mary, Betty, Jean, Helen}

   Set B is a set of girls in this same room who have brown hair.

   B = {Linda, Betty, Helen, Sandra, Nancy}

   (a) What is the number of members of Set G? (4)

   (b) What is the number of members of Set B? (5)

   (c) What is the number of members of $G \cup B$? (7)

   We could not add 4 and 5 to find the number of members of the union of these two sets.

   <u>We can use addition to find the number of elements in the union of two sets only if the two sets have no common members.</u>

Exercise Set 1

Answer each question carefully.

1. Set F = {dog, cow, horse, pig, turkey}

   Set G = {chicken, dog, robin, cat, pig}

   (a) The number of members of Set F is ___ (5)

   (b) The number of members of Set G is ___ (5)

   (c) The number of members of $F \cup G$ is ___ (8)

   Why couldn't we add the number of members of Set F and the number of members of Set G to get the number of members of $F \cup G$? (Because they have 2 members in common)

2. J = {a, b, c, d, e, f, g, h, i}

   K = {j, k, l, m}

   (a) What is the number of members of $J \cup K$? (13)

   (b) Could you add the number of members of Set J and the number of members of Set K to find the answer? (yes)

   Why? (Because the sets have no members in common)

3. M = {f, o, u, r, t, h}

   P = {g, r, a, d, e}

   What is the number of members of $M \cup P$? (10)

4. There are 8 members in Set R.

   There are 5 members in Set S.

   No members of Set R are members of Set S.

   What is the number of members of $R \cup S$? (13)

   How did you find your answer? (I added 8 and 5)

5. Set E has 9 members.

Set F has 7 members, none of which are members of Set E.

What is the number of members in $E \cup F$? [illegible]

6. W = {5, 7, 9, 10, 11, 12, 13, 14}

Y = {13, 14, 15, 16, 17, 18, 19, 20, 21, 22}

The number of members in $W \cup Y$ is [illegible]

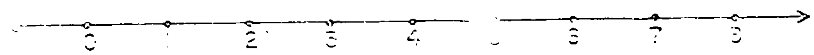
How did you find your answer?

Some children may count the number of different elements in the two sets; others may have added the number of elements in the two sets and then subtracted 2 because the sets have 2 common elements.

ADDITION AND THE NUMBER LINE

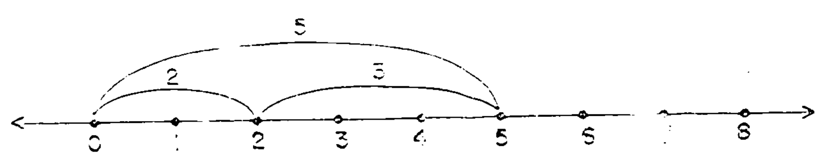
> In "seeing" addition on the number line, recall with the pupils what they learned about the number line in Chapter 2. Both the union of disjoint sets and the number line are helpful in "seeing" the meaning of addition. The use of one does not exclude the use of the other. Some children may be helped by both aids and some may be helped by one and not the other.

Let us call a part of the number line between two points of the line *a line segment*.



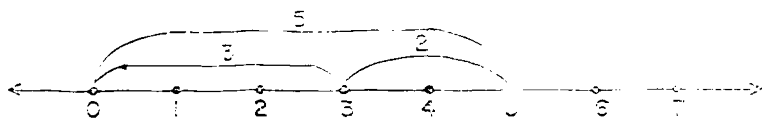
The part of the line between 0 and 1 is a line segment. The points labelled 0 and 1 belong to this line segment. The part of the line between [illegible] and [illegible] is a line segment. We will call such line segments "the line segment from 0 to 1", "the line segment from [illegible] to [illegible]", and [illegible] the same way for others. Name some other line segments on the number line in the diagram. Let us see if we can use the number line and line segments on it to help us picture addition of two numbers.

Does this diagram help you think 2 + 3 = 5?



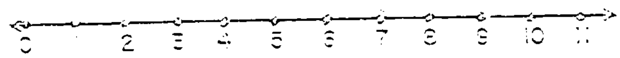
What diagram would you make to help think 3 + 2 = 5?

> Some pupils may suggest the same diagram as for 2 + 3 = 5. Others will want the diagram illustrated below. The objective here is to reach agreement that the same diagram is useful in thinking of both 2 + 3 = 5 and 3 + 2 = 5. If this is established here pupils will have a beginning on the understanding of the commutative property for addition.



Following the discussion the pupils should work individually on Exercise Set 2. Let them compare their answers in a class discussion.

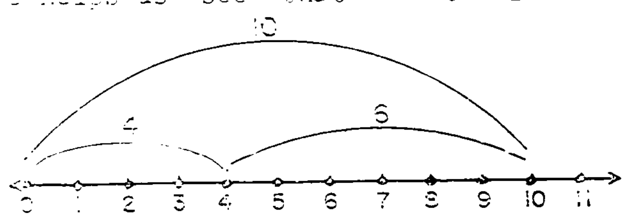
ADDITION AND THE NUMBER LINE



A part of the number line between two points is called a *line segment*. For example the part of the line between the point labelled 3 and the point labelled 6 is called "the line segment from 3 to 6". The points labelled 3 and 6 belong to this line segment. Name some other line segments by looking at the number line above.

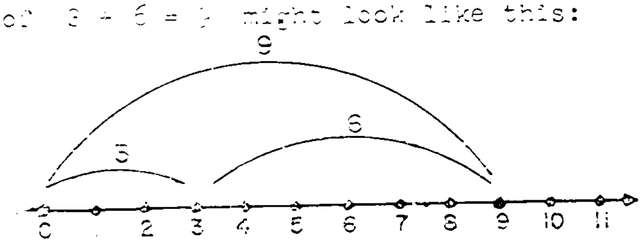
We can use the number line and line segments to help us picture the addition of numbers.

This picture helps us "see" that 4 + 6 = 10



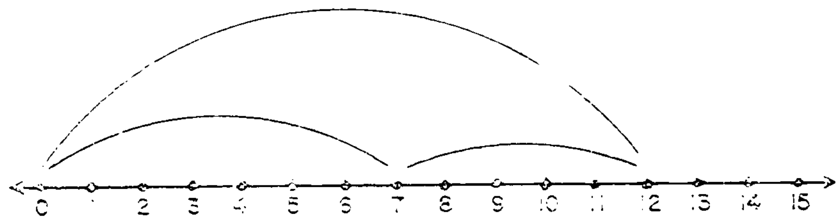
We first draw the curve to picture 4. Next, we draw the curve to picture 6. We start this at 4 and go 6 units to the right. Then we draw the curve from 0 to 10 to picture the sum of 6 and 4.

A picture of 3 + 6 = [illegible] might look like this:

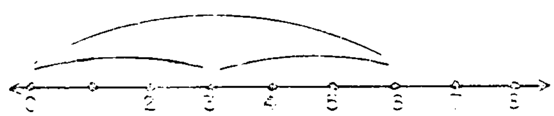


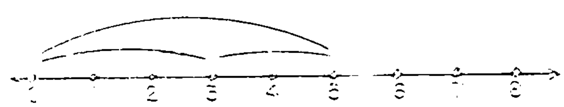
F"O

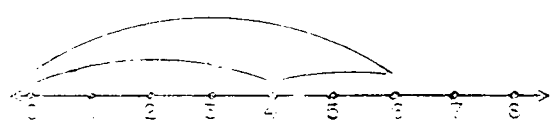
1. What does this picture suggest to you? (7 + 5 = 12)



2. Which of these is a picture of 3 + 2 = 5?

a. 

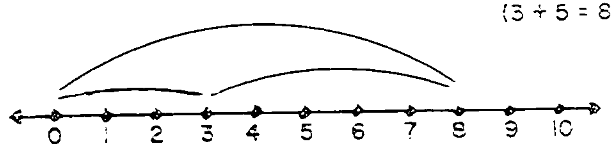




Exercise Set 2

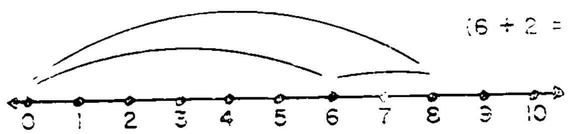
1. Is this a picture of $4 + 3 = 7$ or of $3 + 5 = 8$

(3 + 5 = 8)

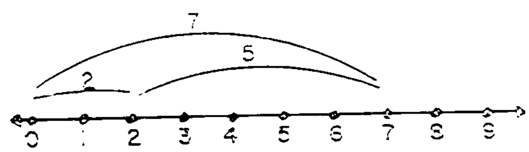


2. What does this picture suggest to you

(6 + 2 = 8)



3. Use a number line to draw a picture of $2 + 5 = 7$



4. Draw a number line picture of each of these.

(a) $7 + 6 = 13$

(b) $5 + 9 = 14$

(c) $4 + [illegible] = 11$

ADDITION AND SUBTRACTION AS OPERATIONS

Objectives: To develop the idea that an operation is a way of thinking about two numbers to produce a unique third number; and, to help children realize that the four basic "processes" of arithmetic (addition, subtraction, multiplication, and division) are operations in this sense

Materials Needed: Felt pieces and a flannel board, sets of objects such as discs

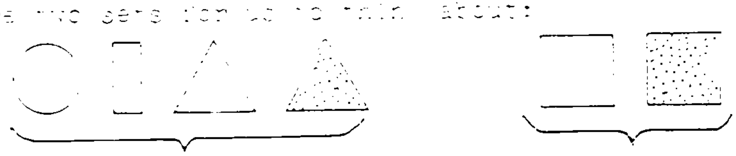
Vocabulary: Operation, pair of numbers, result, order

Teaching Procedures:

> The exploratory material below is written as if a teacher were talking to his class. The answers a teacher wishes to obtain from children in response to questions and suggestions are included in parentheses.
>
> In the development the teacher should make drawings on the chalkboard, use felt pieces, or provide sets of things for use. If he wishes to do so, he may put this exploration in some kind of social context, as two sets of people, books, etc. He should note, however, that Set A and Set B have no members the same but the Set [illegible] is a subset of Set [illegible].

Exploration:

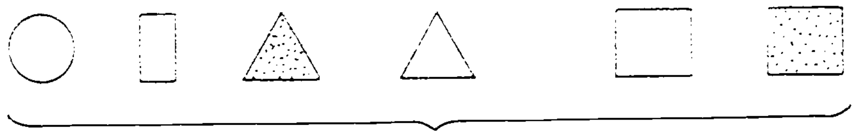
Here are two sets for us to think about:



Set A Set B

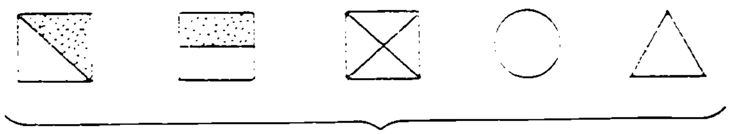
How many members are there in Set A? (4) How many members are there in Set B? (2) Notice that there are no members in Set A that are in Set B.

Show the result of combining these two sets. We will call the result Set C. Set C is the union of Set A and Set B.

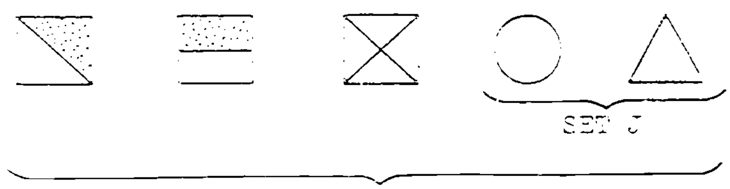


How many members are there in Set C? (6) We can say there are 4 + 2 or 6 members in Set C. 4 + 2 = 6

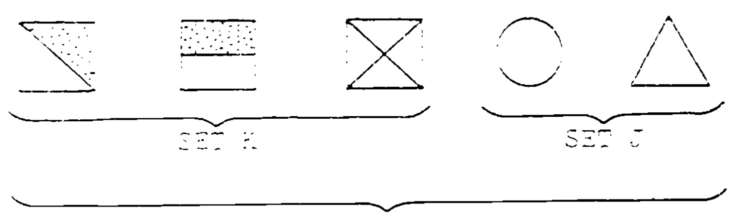
Here is another set for us to study. We will call it Set H.



Within Set H are two members, the circle and the triangle, which we will group together and call Set J. Set J is a part of Set H.



Set J is one part of Set H. There is another part of Set H. We will call the other part Set K. Set K is with Set H.



What is the number of members in Set H? (5) What is the number of members in Set J? (2) What is the number of members in Set K? (3) Our set illustrates that 5 minus 2 is 3 or 5 - 2 = 3. It also illustrates that 5 minus 3 is 2 or 5 - 3 = 2.

When we add, we are operating on numbers. Can you illustrate this with one of the examples we used? What numbers were added? (4 and 2) What was the result? (6)

> If we start with the two sets A and B and get the set C this is called the union set and union is the operation. In starting with the two sets H and J and getting the third set K we are finding the subset of H so that its union with Set J is Set H. We may call this operation partition. It is not meant here to say that operations union and partition on sets are completely analogous to operations of addition and subtraction on numbers. It is meant only to say that we can start with two sets and get a third by means of an operation.

What operation on numbers was used in the second example? (Subtraction) What numbers were operated on? (5 and 2) What was the result? (3)

What are the operations on numbers you have used? (Addition, subtraction) Did you begin in each with two numbers and get one number as a result? (Yes).

> You will notice that with addition and subtraction, we start with a pair of numbers and get a third number which is the result. This is a general way to describe an operation on numbers: we start with a pair of numbers, operate on the numbers, and get a third number as a result.
>
> It should also be emphasized that the result depends on the operation performed. If you begin with 12 and 2 and add, the result is 14; subtract and the result is 10; multiply and the result is 24; divide and the result is 6. The order of writing the numbers operated on is important. For example, if the operation is subtraction and the numbers operated on are 12, 7 it is understood to mean 12 - 7 and not 7 - 12; then 12 is the sum, 7 is one addend, and the other addend is to be found. If the operation is division and the numbers operated on are 6, 2 it is understood to mean 6 ÷ 2 and not 2 ÷ 6, then 6 is the product, 2 is one factor and the other factor is to be found.

ADDITION AND SUBTRACTION AS OPERATIONS

Addition and subtraction are two operations of mathematics. Multiplication and division are also operations. These four operations are called the basic operations.

An operation on two numbers is a way of thinking about two numbers and getting one and only one number. When we think about 9, 5 and get 14, we are adding. We write $9 + 5 = 14$. When we think about 9, 5 and get 4, we are subtracting. We write $9 - 5 = 4$. In subtracting, the order of the two numbers is important. If the numbers are 5, 9 this would mean $5 - 9$. There is no whole number for $5 - 9$.

Exercise Set 3

Number from 1 through 12 on your paper. Write the correct numeral or word to complete this chart. The first exercise is done for you.

	Numbers Operated On	Result	Operation Used
1.	5, 7	12	Addition
2.	9, 3	6	Subtraction
3.	10, 2	12	Addition
4.	10, 2	8	Subtraction
5.	10, 2	5	Division
6.	10, 2	20	Multiplication
7.	5, 4	9	Addition
8.	16, 9	7	Subtraction
9.	9, 6	3	Subtraction
10.	6, 9	15	Addition
11.	11, 7	4	Subtraction
12.	3, 2	1	Subtraction

Now consider with children the other two operations, multiplication and division. Give some illustrations starting with a pair of numbers and the operation, multiplication, and determine the unique third number. Similarly, use illustrations for division, noting that care must be taken in selecting the pair and the order so that the third number is a whole number.

Then discuss each example in the chart below in terms of the two numbers, an operation and the number which is the result.

The mathematician says that these numbers, 12, 7, form an <u>ordered pair</u> meaning that 12, 7 and 7, 12 are different pairs because the numbers are in a different order.

	Operation	Numbers Operate On	Result
(a)	Addition	7, 9	16
(b)	Addition	12, 2	14
(c)	Subtraction	12, [illegible]	5
(d)	Subtraction	12, 2	10
(e)	Subtraction	7, 12	Not a whole number
(f)	Multiplication	6, 4	24
(g)	Multiplication	12, 2	24
(h)	Division	16, 2	8
(i)	Division	2, 16	Not a whole number

Exercise Set 3 may be used now.

TRUE MATHEMATICAL SENTENCES

Objective: To help pupils learn (a) the meaning of mathematical sentences, and (b) that some mathematical sentences are true and that others are not true

Vocabulary: Mathematical sentence, true mathematical sentence, false

Exploration:

In your language work you have learned to write sentences. They may tell you something about your brother or about your city or some action on the playground. Some <u>mathematical sentences</u> tell you something about numbers. $6 + 7 = 13$ tells you that if 6 and 7 are operated on by addition, the result is 13. It is a mathematical sentence about 6, 7, and 13.

> Ask children to give pairs of numbers, the operation to use on them, and the result. For example, someone might give 3 and 7, the operation of addition, and the result as 10. The mathematical sentence about 3, 7, and 10 is $3 + 7 = 10$. If it had been 7 and 3 and the operation addition, the mathematical sentence would be $7 + 3 = 10$.

Often we [illegible] mathematical sentences in which we are not given a [illegible] ber. Suppose I think of a number. What is the number? Of course you cannot tell. But if I tell you that if you add [illegible] and [illegible] you will get the number I am [illegible] you know the number is 10.

> Ask the children to express this as an English sentence. One might say, "The number you are thinking of is [illegible] + 3." Then ask how we can express this same idea as a mathematical sentence. It is [illegible] children will respond with [illegible] + 3 = ___ or [illegible] + [illegible] = [illegible] or [illegible] + 3 = [illegible]. We want to lead them to the idea that a letter can be used to represent a number.

Then we can write, for example, $7 + 3 = n$. Explain that any letter can be used to represent a number but that n is commonly used. Proceed with several similar examples until children become accustomed to mathematical sentences in which a letter is used to represent a number.

When we write the mathematical sentence, $7 + 3 = 10$, we are expressing the idea that both $7 + 3$ and 10 are different ways of naming the number I was thinking of. Think about $8 + 7 = 15$. Is it true that $8 + 7$ and 15 are different names for the same number? (Yes) Are these expressions, [illegible]; [illegible]; and [illegible], names for the same number? (Yes) Are [illegible] and 11 different names for the same number? (No)

Let's think again about mathematical sentences, such as $7 + 3 = n$ or $n + 6 = 9$. We really don't know if these sentences are true or false. If n represents the number 3 in the mathematical sentence, $n + 6 = 9$, then that sentence is true. If n represents any other number, that sentence is false.

Mathematical sentences are just like sentences in English. Sometimes an English sentence is true. Sometimes an English sentence is false (not true). Sometimes we do not know if it is true or false. Suppose we say, "He is absent from school." or "She is going to the library." Are these sentences true or false? We do not know until we know who <u>he</u> or <u>she</u> is. We say that $n + 6 = 9$ is neither true nor false. When n represents the number 3, we can say the sentence is true. If we know that n represents any other number than 3, we know the sentence is false.

Ask children to give some English sentences which are true, some which are false, and some which might be either true or false.

Then, ask them to give some mathematical sentences that are true and which use only the operations of addition and subtraction, and symbols (+, −, =). Discourage the use of >, <, and ≠ at this time. Some possible responses are $11 - 2 = 9$, [illegible], etc.

Ask how they know that these are true mathematical sentences, getting at the idea that they are two different names for the same number, e.g., $11 + 2$ and 13 name the same number.

Now have pupils think about mathematical sentences which are not true or sentences in which you must have more information before you can say that they are true or false. The exploration with your class might go something like this.

What are some mathematical sentences which would not be true? (Various answers e.g., $14 < 6 + 5$) Suppose we change any one number in one of the true mathematical sentences. Is the sentence still true? (No, it is false; e.g., $11 + 2 = 13$ but $10 + 2 = 13$ is not true.)

Is the mathematical sentence, $8 + n = 14$, true or false? (We cannot tell until we know the number n represents) If n represents 12, is it true? (No) Can n represent more than one whole number to make the sentence a true sentence? (No, n must be 6).

As a way of writing this we usually write:

$$8 + n = 14$$
$$n = 6$$

Who can come to the board and write a mathematical sentence to tell what n represents so that this sentence will be true.

$$n + 4 = 12$$

(The pupil should show his work like this:

$$n + 4 = 12$$
$$n = 8$$

Give several other examples so that the children have opportunities to practice writing in this form.

The development in the pupil text, P 73 has been presented as a series of exercises to be used for group discussion <u>following</u> the above exploration.

However, you may wish to give some or all of these to the pupils to do on their own and follow this with a group discussion of the answers.

Note exercises 12 and 13. on page [illegible]. The pupil should begin to recognize that the combining of a set of 6 objects and a set of 5 objects (the two sets have no members in common) into a single set of 11 objects is a model or illustration of $6 + 5 = 11$. Encourage many illustrations using objects and pictures to show sentences such as $5 + 6 = 10$ and $6 - 2 = 3$ are not true mathematical sentences.

In this unit a statement such as "Find n" followed by $n + 7 = 10$ means, "Find n so that $n + 7 = 10$ is a true statement." In some exercises, other language forms (which have the same meaning) such as "What whole number is n so that $n - [illegible] = [illegible]$ is a true mathematical sentence" are used.

Exercise Set [illegible] and Exercise Set [illegible] provide opportunity for children to work on their own.

TRUE MATHEMATICAL SENTENCES

A sentence which tells us something about numbers is a mathematical sentence.

Statements like those in Box A are called mathematical sentences. A mathematical sentence can be true or false.

A
$6 + 4 = 10$
$9 - 3 = 2$
$0 + 8 = 8$

1. (a) Is it true that $6 + 4 = 10$? yes
   (b) Is it true that $9 - 3 = 2$? no
   that $7 + 6 \neq 12$? yes
   that $10 - 10 \neq 10 - 9$? yes
2. (a) Are $6 + 4$ and 10 different names for the same number? yes
   (b) Are these names for the same number: $6 + 3$, $11 - 2$, $9 + 0$? yes
   (c) The base ten numeral for $9 - 3$ is 6. What is the base ten numeral for: $17 - 3$; $12 + 5$; $9 - 0$?
3. Write some other mathematical sentences.
4. Mary said, "I'm thinking of a number. The number is the result of adding 5 and 3. Of what number am I thinking?"
5. Bob said, "Let's call Mary's number n. Then, n is the result of adding 5 and 3. So, $n = 8$. Now, if $n = 8$, then $5 + 3 = n$."

   Statements like those in Box B are also called mathematical sentences.

B
$3 + 8 = n$
$12 - 7 = n$
$n - 4 = 7$
$4 - n = 1$

6. (a) If $n = 5$, is $2 + 3 = n$? (yes)
   (b) If $n = 9$, is $n - 5 = 4$? (yes)
   (c) If $n = 5$, is $n - 5 = 1$? (no)
   (d) If $n = 7$, is $3 + 5 = n$? (no)

7. (a) If $n = 4$, is $10 - n = 6$? (yes)
   (b) If $n = 7$, is $n + 3 = 10$? (yes)
   (c) If $n = 9$, is $n + 3 = 10$? (no)
   (d) If $n = 7$, is $10 - n = 2$? (no)

8. (a) If $n = 8 + 7$, is $9 + 6 = n$? (yes)
   (b) If $n = 30 - 20$, is $15 + n = 25$? (yes)
   (c) If $n = 100 + 300$, is $n - 200 = 200$? (yes)

9. What is the base ten numeral for n in each of (a), (b), (c), in Exercise 8. a (15) b (10) c (400)

10. (a) If $n = 7$, is $11 - 3 \neq n$ or is $11 - 3 = n$? ($11 - 3 \neq n$)
    (b) If $n = 12$, is $17 - 5 \neq n$ or is $17 - 5 = n$? ($17 - 5 = n$)
    (c) If $n = 1$, is $12 - n \neq 11$ or is $12 - n = 11$? ($12 - n = 11$)

11. If $n = 7$, which of these are different names for n? (a, d)

    (a) $12 - 5$ (b) $19 - 13$ (c) $10 - 7$ (d) $27 - 20$

12. Joan said, "5 + 5 = 11 is a mathematical sentence. It is true." Do you think Joan is right? (yes) Draw a model or picture to show why you think as you do.
See Teacher's Commentary page 137

13. Bill said, "6 - 2 = 3 is a mathematical sentence. It is true." Do you agree with Bill? No Draw a picture or model to show why you think as you do.
(6 - 2 = 3 is a mathematical sentence but it is not a true sentence)

> The mathematical sentence 6 + 7 = 13 is true. It is true because 6 + 7 and 13 are different names for the same number.
>
> The mathematical sentence 6 + 7 = 12 is not true. It is not true because 6 + 7 and 12 are not different names for the same number.

Exercise Set 4

1. Some of these mathematical sentences are true. Write on your paper the letter of each mathematical sentence that is true. (a, c, f, g, l)

(a) 2 + 1 = 3	(g) 17 - 11 = 6
(b) 6 + 1 = 1[illegible]	(h) 19 - 12 = 5
(c) 9 + 14 = 23	(i) 26 + 21 = 74
(d) 11 - 1 = 11	(j) 33 - 21 = 2
(e) 13 - 8 = [illegible]	(k) 62 + 6 = 122
(f) 9 - 3 = 6	(l) 78 - 62 = 16

2. If n = [illegible] which of these mathematical sentences are true? [illegible]

(a) $1 +$ [illegible] (b) [illegible] $= 5$ (c) [illegible] $- 2 = 3$

3. Suppose you were asked what number must n represent so that $8 + 5 = n$ is a true mathematical sentence.

(a) If you said "n is 13," would you be correct? yes

(b) If you said "n is 12," would you be correct? no

(c) Would it be correct to say, "If $n = 13$, then $8 + 5 = n$." yes

4. If n represents the number 3, which of these sentences are true? [illegible]

(a) $2 + n = 5$ (d) $9 - n = 5$

(b) $8 + n = 10$ (e) $6 - n = 3$

(c) $n - 3 = 6$ (f) $3 - n = 0$

5. If $n = 7$, which of these are true mathematical sentences? a, b, c, d

(a) $n + 9 = 16$ (c) $9 - n = 2$ (e) $5 + n = 12$

(b) $n - 6 = 1$ (d) $4 + 3 = n$ (f) $12 - n = 6$

6. (a) What number is represented by n so that $8 + 4 = n$ is a true mathematical sentence? (12)

(b) What number is represented by n so that $2 + n = 11$ is a true mathematical sentence? (9)

7. What number is represented by n so that $3 + n = 9$ is a true sentence? You may use the form in Box A to write your answer.

A
$3 + n = 9$
$n = 6$

8. What number is n so that $2 + n = 7$ is a true sentence? 5 Write your answer in the same form that you used in exercise 7.

Exercise Set 1

Copy the numerals 1-15 on your paper. Next to each, write the correct words, numerals and mathematical sentences to complete this chart. The first one is done for you.

	Numbers Operated On	Result	Operation Used	Mathematical Sentence
1.	12, 9	3	Subtraction	12 − 9 = 3
2.	18, 9	[illegible]	Subtraction	18 − 9 = 9
3.	6, 3	[illegible]	Addition	6 + 3 = 9
4.	5, 8	40	Multiplication	5 × 8 = 40
5.	18, 3	6	Division	18 ÷ 3 = 6
6.	6, [illegible]	13	Addition	6 + m = 13
7.	[illegible], 7	6	Subtraction	b − 7 = 6
8.	3, [illegible]	[illegible]	Addition	3 + [illegible] = 10
9.	14, [illegible]	[illegible]	Subtraction	14 − t = 8
10.	12, 7	5	Subtraction	12 − 7 = 5
11.	15, 9	[illegible]	Subtraction	15 − 9 = 6
12.	3, 14	17	Addition	3 + 14 = 17
13.	12, 4	3	Division	12 ÷ 4 = 3
14.	[illegible], 3	5	Multiplication	5 × 3 = 15
15.	6, 2	4	Subtraction	6 − 2 = 4

THINKING ABOUT ADDITION FACTS

Objective: [illegible]

Vocabulary: Addend, sum, plus, column, row

Teaching Procedures:

The amount of review of the addition facts needed by pupils will vary. While some pupils need less drill than others, every pupil should study carefully the relationships which are emphasized in this section. It is expected that purposeful drill wrok for some pupils will need to be provided by the teacher.

Page [illegible] in the pupil's book should be done by teacher and pupils working together. It is important that the pupils state the addition facts in different language forms as indicated on the top of that page. The words, addend and sum should be used repeatedly. The teacher may find it helpful to duplicate the addition chart so each pupil will not need to copy it.

THINKING ABOUT ADDITION FACTS

You show addition like this:

9 + 5 = 14	9 +5 14	9 +	5 =	14
		addend	addend	sum

You read addition like this:

9 and 5 are 14, or 5 added to 9 is 14, or 9 plus 5 is 14, or 9 plus 5 equals 14, or 9 + 5 is equal to 14.

In each of these examples tell which numbers are addends and which is the sum.

(a) 8 + 5 = 13

(b) 8 + 9 = 17

(c) 10 + 20 = 30

(d) $\begin{array}{r} 31 \\ +45 \\ \hline 76 \end{array}$

(e) $\begin{array}{r} 23 \\ +64 \\ \hline 87 \end{array}$

You may know all the addition facts. This will help you in completing the chart on the next page. It is called an addition chart. To begin, you add to the number 0, which is in the left column, each of the numbers in the top row. Write the sum under the number which was added to zero. Add, 0 + 0 = 0 and write 0 under zero; 0 + 1 = 1, so write 1 under 1, and so on. Make a copy of the chart on the next page and finish it.

Addition Chart

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	[illegible]	5	[illegible]	7	8	9
1	[illegible]	2	3	[illegible]	5	[illegible]	[illegible]	8	9	10
2	2	3	[illegible]	5	[illegible]	[illegible]	8	[illegible]	10	11
3	3	4	5	[illegible]	[illegible]	[illegible]	[illegible]	10	11	12
4	4	5	6	[illegible]	8	[illegible]	[illegible]	[illegible]	12	13
5	5	6	[illegible]	8	[illegible]	[illegible]	[illegible]	12	13	14
6	6	[illegible]	8	[illegible]	10	[illegible]	12	13	14	15
7	[illegible]	8	9	10	[illegible]	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

Children may now use the chart. Suggestions for discussions follow:

Now that you have the chart, how can you use it. (We can use it to find the sum of any two numbers in the set 0 through 9.)

Let us examine the chart and list some relationships which you noticed as you made the chart. As we list these, perhaps you will find other relationships.

(a) Many sums are the same and they are all in a slanting line. (After many relationships are listed, the teacher may help pupils to recognize that if one addend is increased by 1 and the other decreased by 1, the sum is unchanged. This may be illustrated using materials or pictures.)

(b) The first row we wrote across the top is the same as the one above it and the first column we wrote down the side is the same as the one to its left. (This is an important observation because it leads to the property of zero as an addend in addition and subtraction. As soon as all observations are listed, the teacher should ask questions such as: "What is 0 + 9? 0 + 5, 0 + [illegible]",

(c) The chart shows that 7 and 5 have the same sum as 5 and 7, etc. (Use many similar statements to emphasize this idea, that the order of addends may be changed without changing the sum. This is studied carefully in the next section.

(d) Each sum in a row is one more than the sum it follows; each sum in a column is one more than the sum above it. (The teacher may ask, "Why?" after each relationship is listed.)

[illegible] as they will [illegible] for a few days. [illegible] in mind that [illegible] practice on [illegible] will [illegible] The [illegible] pupils who have [illegible] and [illegible] for additional practice. [illegible] and use [illegible] facts (without answers) and [illegible] for their [illegible] every day. [illegible]

[illegible] exercise [illegible] now. It is designed to help pupils [illegible] facts and [illegible] 11 [illegible] facts.

[illegible] are some approaches to [illegible]:

1) To add [illegible] to any number, you can add 10 [illegible] subtract 1 because 10 − 1 is another name for 9.

(2) [illegible] pupils [illegible] statements such as "Because [illegible] 10 [illegible] = 11."

(3) Help pupils give reasons for their answers. For: "Because [illegible] = 12, then [illegible] = 13." The teacher cannot expect the answer in this form, "If one addend is unchanged and the other increased by 1, the sum is increased by 1." However, we can [illegible] it in these words and we can strive for better and better wording for the understanding of the pupils.

Use these relationships as ways of thinking about numbers. As a final outcome pupils should know the addition facts from memory.

Exercise Set 6

Addition of numbers as in the box may be done quickly if you know the addition facts. You should be able to recall all addition facts given in the addition chart. Here are a few ideas in case you forgot some facts.

A	
346	901
+195	+648

1. Complete these statements:

   (a) Because $7 + 7 = 14$, $7 + 8 =$ (15)

   (b) Because $6 + 6 = 12$, $6 + 7 =$ (13)

   (c) Because $5 + 5 = 10$, $6 + 5 =$ (11)

   (d) Because $8 + 8 = 16$, $8 + 7 =$ (15)

2. (a) You know $6 + 8 = 14$. How do you find $6 + 9$?
   (6+9 is one more than 6+8)

   (b) You know $9 + 4 = 13$. How do you find $9 + 5$?
   (9+5 is one more than 9+4)

   (c) You know $7 + 9 = 16$. How do you find $8 + 9$?
   (8+9 is one more than 7+9)

3. Complete these statements:

   (a) Because $10 + 9 = 19$, $9 + 9 =$ (18).

   (b) Because $10 + 8 = 18$, $9 + 8 =$ (17).

   (c) Because $7 + 10 = 17$, $7 + 9 =$ (16).

   (d) State a way to add 9 to any number.
   (Add 10 and then subtract 1)

4. Complete these statements:

(a) 0 + 5 = 5

(b) 9 + 0 = 9

(c) 6 + 0 = 6

If you add 0 to any number, what is the sum?
The number with which you started.

5. Complete these statements:

(a) 9 + 1 = 10

(b) 7 + 1 = 8

(c) 1 + 8 = 9

If you add 1 to any number, what is the sum?
One more than the number.

6. Look at the addition chart you made. Name the numbers in each of the sets described below.

(a) The members of set A are found by adding 9 to each of the numbers in the set {0, 1, 2, 3, ..., 9} {9, 10, 11, 12, 13, 14, 15, 16, 17, 18}

(b) The members of set B are obtained by adding 7 to each of the numbers in the set {0, 1, 2, 3, ... 9} {7, 8, 9, 10, 11, 12, 13, 14, 15, 16}

(c) The members of set C are obtained by adding 6 to each of the numbers in the set {0, 1, 2, 3, ..., 9} {6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

7. BRAINTWISTER: Use your answer to exercise 6 to find:

(a) $A \cap B$ (d) $A \cup B$

(b) $A \cap C$ (e) $A \cup C$

(c) $B \cap C$ (f) $B \cup C$

a $A \cap B$ $\{9, 10, 11, 12, 13, 14, 15, 16\}$
b $A \cap C$ $\{9, 10, 11, 12, 13, 14, 15\}$
c $B \cap C$ $\{7, 8, 9, 10, 11, 12, 13, 14, 15\}$
d $A \cup B$ $\{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$
e $A \cup C$ $\{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$
f $B \cup C$ $\{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$

8. BRAINTWISTER:

(a) Use your answer to exercise 6 to find

$(A \cap B) \cap (B \cap C)$. $\{9, 10, 11, 12, 13, 14, 15\}$

(b) Your answer for Exercise (a) is the same as which answer of Exercise 7? (b)

THE COMMUTATIVE PROPERTY FOR ADDITION

Objective: To help children understand that in addition, the order of addends may be changed without changing the result ($3 + 5 = 5 + 3$ or, in general, $a + b = b + a$, where a and b are any whole numbers.

Materials Needed: Blocks or other objects, coins (a dime and a quarter)

Vocabulary: Commutative property for addition

Teaching Procedures:

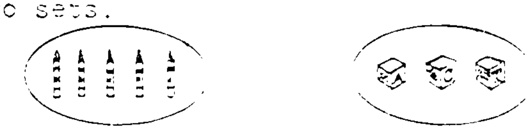
It is important that pupils understand and use the commutative property for addition. This property is sometimes called the <u>order property</u> for addition; however, the term commutative property for addition is used here.

The following development requires the use of examples of combining activities which the teacher may elicit from the class.

The teacher may want to use materials to demonstrate the effect of order and grouping in both nonnumerical and numerical situations. Use objects such as crayons or strips of cardboard, pencils, strings of beads, etc.

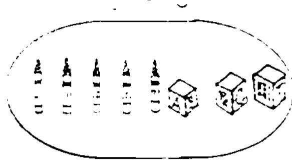
The lesson may be introduced by putting two sets of objects, for example, crayons and blocks, together in two different ways. The illustrations indicate the way you might use the two sets.

1.

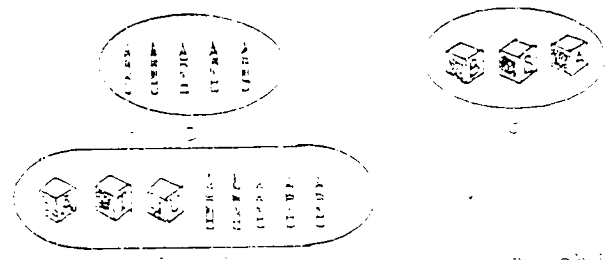


5 3

5 + 3



8 objects in the union



The important observation to make is that the order of putting the sets together does not affect the result. In either case the union of the 2 disjoint sets consists of 8 members, 5 crayons and 3 blocks. Use similar examples to strengthen the concept that the sum of two numbers is not affected by the order in which the two numbers are added.

Nonnumerical and numerical examples of combining things and observing the results are described on P 15. These should be discussed with the class.

The important outcome of this lesson is to see that the order of adding two whole numbers does not change the sum. Use many examples, asking "Is it true that 7 + 4 = 4 + 7, 0 + 6 = 6 + 0, 12 + 21 = 21 + 12, 96 + 69 = 69 + 96, etc.?" There should be many examples until pupils understand, generalize, and freely use the property. They should be able to give examples and recognize that $a + b = b + a$ is a statement about all pairs of whole numbers.

Experimentation should be performed to determine if this property applies to subtraction. In finding out if $5 - 2 = 2 - 5$ you can help pupils see that $3 = 5 - 2$, but they know no whole number to rename $2 - 5$. The teacher may want to illustrate this by putting 2 pencils in a box and trying to take out 5 and then putting 5 pencils in a box and trying to take out 2.

There is no intent here to discuss "negative numbers." A teacher may look forward, however, with the children to a time when they will use numbers which will allow them to find the number n so $2 - 5 = n$ is a true mathematical sentence.

Assign Exercise Set [illegible] after [illegible] development and P[illegible] have been discussed.

THE COMMUTATIVE PROPERTY FOR ADDITION

1. Pretend you have a dime and a quarter. You are paying for a 35 c book with this quarter and dime. Would the order of giving the coins to the clerk make a difference in what you paid him? (no)

2. Use the letters "O" and "N" to make two words. What are the words? (no) (on) Did the order of the letters change the result? (yes)

3. Is the result the same for each addition in (a)? (yes) in (b)? (yes) in (c)? (yes)

(a)	(b)	(c)
7 + 5, 5 + 7	30 + 40, 40 + 30	0 + 891, 891 + 0

How are the sums different in (a)? in (b)? in (c)?
The order of the addends is different.

4. (a) Is 400 + 500 = 500 + 400? (yes)
(b) Is 692 + 0 = 0 + 692? (yes)
(c) Is 1,000,000 + 0 = 0 + 1,000,000? (yes)
(d) If n is a whole number, is n + 10 = 10 + n? (yes)

ADDITION IS A COMMUTATIVE OPERATION

For example, 3 + 5 = 8
5 + 3 = 8

The sum is the same even if the order of the addends is changed. So we can write

3 + 5 = 5 + 3.

Exercise Set I

1. Is 5 - 2 = 2 - 5? (no) Is 9 - 7 = 7 - 9? (no) Do you know what number 2 - 5 is? (no) Does the commutative property seem to hold for subtraction? (no)

2. A teacher was reading to a class, "What number is represented by n so that 637 + 596 = n?" Jim did not hear the "637" so he wrote, 596 + _____ = n. He asked the teacher to tell him the first addend. He then wrote, 596 + 637 = n. Will his result be the same as the pupil who wrote 637 + 596 = n? (yes) Why? Addition is a commutative operation.

3. (a) Is 12 - 12 = 12 - 12? (yes)
   (b) Is 10 - 20 = 20 - 10? (no)
   (c) If n = 10, is n - n = n - n? (yes)
   (d) If n = 20, is n - n = n - n? (yes)
   Tell what number n represents so that n - 6 = 6 - n. (6)

4. Which of the following are true mathematical sentences? a, b, c, e
   (a) 17 - 11 = 11 - 17
   (b) 305 + 401 = 200 + 40[illegible]
   (c) [illegible]
   (d) 1.20[illegible] + 3.011 = 3.006 + [illegible]
   (e) 14 - 11 = 11 - 1[illegible]
   (f) [illegible]

5. Which mathematical sentences in exercise 4 illustrate the commutative property? (a, e)

6. BRAINTWISTER: (a) Write the numerals for whole numbers (if possible) to complete this chart.

Numbers Operated On	Result	Operation Used
15, 7	(22)	Addition
7, 15	(22)	Addition
15, 7	(8)	Subtraction

(b) Is there a whole number for each blank? (yes) If not, why?

7. BRAINTWISTER: (a) What whole number is represented by n so that $0 + n = n + 0$? (any whole number)

(b) What whole number is represented by n so that $12 - n = n - 12$ (12)

(c) What whole numbers are represented by x and y so that $x - y = y - x$? (any whole numbers)

THINKING ABOUT SUBTRACTION FACTS

Objective: To help pupils learn the subtraction facts and to understand subtraction as an operation for finding the unknown addend when the sum and one addend are known

Vocabulary: Known addend, unknown addend

Teaching Procedures:

Particular attention is given in this section to mathematical sentences, such as $9 - 5 = n$. Such sentences are read, "9 minus 5 equals n" "9 subtract 5 equals n" "5 subtracted from 9 equals n." Notice that "take away" is not used yet. Regardless of the way sentences like these are read, the relationship between addition and subtraction is emphasized in this unit by thinking, "What added to 5 is 9?" In this sentence 9 is the sum, 5 is the known addend, and the result is the unknown addend. By remembering that $5 + n = 9$ the pupil is able to determine the unknown addend, represented by n. Thus if a child knows the addition fact he has the background for learning the subtraction facts.

Special attention is also given to help pupils with the new vocabulary instead of the commonly used vocabulary of minuend, subtrahend, and difference or remainder. As teachers, in thinking about addition and subtraction, we [illegible] addition is considered as the [illegible] operation. The relation of subtraction to addition is an effort to help the pupil reduce the amount of memorization, the number of new words needed, and to help him see many relationships that would not otherwise be [illegible] encouraged to [illegible] 12 [illegible] to find [illegible] the pupil who thinks [illegible] should not be forced to change.

[illegible] first focus attention [illegible] new way [illegible] related to [illegible] in [illegible] following

In the mathematical sentence, [illegible] + [illegible] = 13, what are the addends? (6 and 7) What is the sum? (13) In the mathematical sentence, 13 − 6 = 7, we will call 13 the sum; 6 and 7 are called addends. What are the addends in 3 + 8 = 11; in 11 − 3 = 8; in 11 − 8 = 3? (3, 8; 3, 8; 8, 3)

> Such examples as these might help.
> $[illegible] + n = 17$; $n + 7 = 15$; $15 - 6 = n$;
> $11 - [illegible] = n$; $n = 14 - 8$; $[illegible] + n = 14$;
> $n = 16 - 7$; $n + 8 = 11$

Suppose our mathematical sentence is $11 + n = 12$. What are the addends? (One addend is 11. The other addend is the number represented by n.) What is the sum? (12) What is the sum in $n = 12 - 11$? (12) What are the addends? (One addend is the number represented by n. The other addend is 11.) In $n + 11 = 12$ and $n = 12 - 11$, are the numbers, 11 and the number represented by n, the addends in both mathematical sentences? (Yes) Which is the known addend? (11) Which is the unknown addend? (The number represented by n.)

> Select other similar examples and ask questions like those in the above discussion until pupils can select the addends and the sum.
>
> Pupils should then complete the Subtraction Chart P [illegible]. Directions for completing the chart follow. Each number in the top row is to be subtracted from each of those in the first column. The result of 0 − 0 is written under the 0 in the top row. The result of 1 − 0 is written to the right of the 1, and so on. Then 1 in the top row is to be subtracted from each number in the first column. The first of these subtractions is 0 − 1. Ask pupils if this is possible. The chart is only partially completed because there is no result for certain subtractions, such as 0 − 1, [illegible] − [illegible], [illegible] − [illegible], and [illegible] − [illegible], when we use only whole numbers. Write 0 − 1 = n on the board. Ask the pupils to identify the sum (0) and the addends (1 and n). Help them see there is no whole number which n can represent so that 1 + n = 0. Do the same with some others such as [illegible] and [illegible].

This should help reinforce understanding of terms "addend" and "sum" in subtraction. There will be more work on this in a later chapter.

THINKING ABOUT SUBTRACTION FACTS

Subtraction is the operation of finding the unknown addend if we know the sum and one addend. For example, if $8 + n = 12$ then 8 is one addend and n is the unknown addend. We subtract 8 from 12 to find the number that n represents.

You show subtraction like this	The names of the parts in a subtraction sentence are:
$9 - 5 = 4$ $\quad \begin{array}{r} 9 \\ -5 \\ \hline 4 \end{array}$	$\underset{\text{sum}}{9} - \underset{\text{addend}}{5} = \underset{\text{addend}}{4}$

You read subtraction like this: 9 minus 5 equals 4 or 5 subtracted from 9 is 4.

In each of these examples tell which numbers are addends and which number is the sum.

(a) $15 - 9 = 6$ (c) $n + 5 = 13$ (e) $17 - n = 8$

(b) $9 = 13 - 4$ (d) $3 + n = 12$ (f) $14 - 8 = n$

You know many subtraction facts. We will call the chart in which we list them a subtraction chart. Copy the chart on the next page and fill it in.

You begin with zero in the top row. Subtract it from each number in the first column. For example, $0 - 0 = 0$; place the result, 0, [illegible] the right of the zero that is in the first column, $1 - 0 = 1$; [illegible] 1 to the right of the 1, and so on. Go on. To get your [illegible] think [illegible] for $5 - [illegible]$: "What added to 5 is [illegible] [illegible] the numbers in the first column is a sum. All the other numbers in the chart are addends.

Subtraction Chart

-	0	1	2	3	4	5	6	7	8	9
0	0									
1	1	(0)								
2	2	(1)	(0)							
3	(3)	(2)	(1)	(0)						
4	(4)	(3)	(2)	(1)	(0)					
5	5	(4)	(3)	(2)	(1)	(0)				
6	6	5	4	(3)	(2)	(1)	(0)			
7	7	6	5	4	(3)	(2)	(1)	(0)		
8	(8)	7	6	5	(4)	(3)	(2)	(1)	(0)	
9	(9)	8	7	(6)	(5)	(4)	(3)	(2)	(1)	(0)

After the subtraction chart is completed, certain relationships should be emphasized. Some suggestions for class discussion follow:

Here are a few questions we should investigate:

1. Is $4 - 3 = 3 - 4$? (No) Is $6 - 2 = 2 - 6$? (No)

2. Why can't you find the result for $3 - 5$? (There is no whole number which added to 5 is 3.) (Pupils may try to find the number in a set which, when combined with a set of 5 members, results in a set of 3 members.)

3. How do you know that $7 - 4 = 3$? (Accept as the best answer: "I know it is 3 because $4 + 3 = 7$ or $3 + 4 = 7$")

4. Is the known addend or the unknown addend ever larger than the sum? (No) Is either ever equal to the sum? (Yes, e.g., $7 - 7 = 0$ or $3 - 3 = 0$, or $7 - 0 = 7$, or $3 - 0 = 3$.)

Now use Exercise Set 8 and Exercise Set 9. Both groups of exercises should be discussed in class after pupils have worked on them independently.

If you have children write their answer, you should not expect them to write answers in perfect language. Some fast learners may be challenged to attempt the formulation of perfect written answers. This assignment may be made in lieu of unneeded practice of the facts. The Braintwisters of Exercise Set 8 may also be assigned to the fast learners.

Exercise Set [illegible] should be done orally. Its purpose is to help pupils study systematically the relation between addition and subtraction.

You may find the following questions helpful in using the chart on P[illegible].

Look at the chart on page 8 by. Have you ever seen it before? (Yes) How did you use it? (To find the sum of any two numbers in the set 0 through 9.) Where do you find the addends, 6 and 7 for $6 + 7 = 13$? (6 is in the left column and 7 is in the top row.) Where do you find the sum 13? (In space opposite the 6 that is in the left column and below the 7 that is in the top row) How can you find $13 - 6 = n$? Think: $6 + n = 13$. You can find 6 in the lefthand column. Then find 13 in the table in the same row in which you have found the 6. The unknown addend is in the top row above 13. It is 7.

Answers to the above questions are difficult for the children to phrase, but they can point out how to find $12 - 7$, $9 - 4$ and other subtractions from the table by thinking $7 + n = 12$ and $4 + n = 9$

It is important to think of subtractions like $12 - 7$ as, "What added to 7 is 12?" <u>Unless this is done</u>, pupils may not recognize relationship of subtraction to addition.

Many examples like 4 to 9 in Exercise 9, should be stated. Some pupils may write other examples and exchange with their neighbors.

<u>Pupils who have not memorized the addition facts will have difficulty</u>. Provide interesting drill, use flash cards, have children make their own set of flash cards, play authentic games which strengthen skills. Use short quizzes, provide interesting worksheets, and use any other methods you have found effective in helping children memorize the operation facts. Continue to review the facts as needed. Pupils who need practice may be asked to write statements like exercise 4 to 6 in Exercise Set 9 to write other generalizations about the facts.

Practice in writing the facts horizontally, e.g., $9 + 5 = 14$ and $13 - 6 = 7$ should be emphasized because this form will be used often in the following chapters.

Exercise Set 8

Use your subtraction chart to help you answer these questions.

1. Why is the first column which you wrote the same as the column to its left? (A number is not changed when zero is subtracted from it. $n-0=n$)

2. Why do the numbers in each row decrease to zero? (As I go across the number I subtract is one greater than the previous number.)

3. Why do the numbers in each column increase? (The addend is unchanged and the sum is increased by 1)

4. Why is part of the chart empty? (I cannot subtract a larger number from a smaller number using the set of whole numbers.)

5. Study the subtraction chart you finished. Write these sets of numbers:

   (a) The members of set X are the only possible numbers that can be addends if the sum is 3. Set X = {0, 1, 2, 3}

   (b) The members of set Y are the only possible numbers that can be addends if the sum is 7. Set Y = {0, 1, 2, 3, 4, 5, 6, 7}

   (c) The members of set Z are the only possible numbers that can be addends if the sum is 9. Set Z = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

6. BRAINTWISTER: Use your answer to Exercise 5 to find:

   (a) $X \cap Y$ (c) $Y \cap Z$ (e) $X \cup Z$

   (b) $X \cap Z$ (d) $X \cup Y$ (f) $Y \cup Z$

a. {0, 1, 2, 3}
b. {0, 1, 2, 3}
c. {0, 1, 2, 3, 4, 5, 6, 7}
d. {0, 1, 2, 3, 4, 5, 6, 7}
e. {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
f. {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

7. BRAINTWISTER: Use your answer to Exercise 6 to find $(X \cap Y) \cap (Y \cap Z)$. {0, 1, 2, 3}

Exercise Set 9

Subtraction of numbers as in Box A may be done quickly and accurately if you know the addition facts. You should be able to recall all unknown addends in the addition chart.

A	
892	310
-375	-184

1. Tell how to locate the unknown addend for $9 - 4 = n$; $11 - 7 = n$; $16 - 8 = n$; $8 - 0 = n$; $10 - 3 = n$; $12 - 9 = n$.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

(To locate the unknown addend for $13 - 6 = n$, think $6 + n = 13$. Then find 6 in the left-hand column of the table. Look to the right for 13. The unknown addend 7 is at the top of this column.)

2. If you know that $9 + 4 = 13$ what subtraction facts do you know?
   ($13 - 9 = 4$) ($13 - 4 = 9$)

3. If you subtract 0 from any number, what is the result?
   (The result is that number.)

4. Here are some ideas in case you forget some facts.

   Complete these statements:

   (a) Because 16 [illegible], 16 − [illegible] = 9.

   (b) Because 1[illegible] − [illegible], 13 − 4 = 9.

   (c) Because 1[illegible] − [illegible], 1[illegible] − [illegible] = 6.

   (d) Because [illegible] − [illegible], 11 − 5 = ____

5. (a) You know that $14 - 7 = 7$. How do you find $14 - 8$?
(The result is 1 less than $14 - 7$.)

(b) You know that $12 - 6 = 6$. How do you find $12 - 5$?
(The result is 1 more than $12 - 6$)

(c) You know that $15 - 9 = 6$. How do you find $15 - 8$?
(The result is 1 more than $15 - 9$)

6. Complete these statements:

(a) Because $18 - 10 = 8$, $18 - 9 =$ (9).

(b) Because $16 - 10 = 6$, $16 - 9 =$ (7).

(c) Because $13 - 10 = 3$, $13 - 9 =$ (4).

(d) State a way to subtract 9 from any number.
(Subtract 10 from the number and add 1 to the result)

7. (a) Because $13 - 8 = 5$, $13 - 5 =$ (8).

(b) Because $11 - 4 = 7$, $11 - 7 =$ (4).

(c) Because $15 - 6 = 9$, $15 - 9 =$ (6).

8. If you subtract 1 from any counting number, what is the result? (One less than that counting number)

9. Using only the numbers 12, 5, and 7, state two additions and two subtractions. ($5+7=12$ $12-5=7$ $7+5=12$ $12-7=5$)

10. BRAINTWISTER: The result of the operation of subtraction on a pair of numbers is 7. Write five of these pairs.
(Any pair in which the result is 7 such as:

$7-0=7$ $19-12=7$
$15-8=7$ $143-136=7$
$13-6=7$ $201-194=7$)
$16-9=7$

MATHEMATICAL SENTENCES USING THE NUMBER LINE

Objective: To help pupils recognize order relationships and to understand mathematical sentences.

Teaching Procedures:

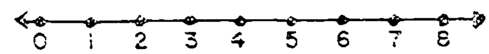
> Material on P. 91 is a review of the meaning of $<$ and $>$. These are the symbols for the order relationships <u>less than</u> and <u>greater than</u>. Have the pupils read the material on P. 45. Write on the board some sentences such as $5 > 3$. Also draw a number line on the board. First make it clear that, for example, 5 is greater than 3 (or 3 is less than 5), since there is a number 2 which added to 3 gives 5 as the sum. (In general if a and b represent any numbers then $b > a$ if there is some number c such that $a + c = b$. If $b > a$, then $a < b$). Then use the number line to help the pupils see that the greater of two numbers is the one farther to the right on the number line. Emphasize that one number is greater than another because a number can be added to the smaller to get the greater.

Is there a number that we can add to 3 to get 5 as the sum. (Yes, the number 2) Which of the numbers, 5 and 3, is the greater? ($5 > 3$, and $3 < 5$) Which number, 5 or 3, is farther to the right on the number line? (5 is to the right of 3. 3 is to the left of 5)

> Use other examples as needed to help the pupils see that if $6 + 3 = 9$ for example, that $9 > 6$ and $9 > 3$ and that 9 is farther to the right than 6 and 3 on the number line.
>
> Use other examples such as [illegible] $> $ [illegible], $5 < 10$, $8 < 5$, and ask pupils if they are true. Ask pupils to make similar statements and tell if they are true or not true. You may now have them complete Exercise Set 10.
>
> In completing an exercise such as example 7 (1200 + 1000) (1200 + 2000), pupils should study it carefully and avoid computation. Here it can be emphasized that 1200 + 1000 is less than 1200 + 2000 without finding the result of each operation.

MATHEMATICAL SENTENCES USING THE NUMBER LINE

Using the Number Line



You remember that this is called a number line.

We can add 2 and 3 to get 5. "5 is greater than 2" and "5 is greater than 3". We write $5 > 2$ and $5 > 3$.

Since $5 > 2$, 5 is to the right of 2 on the number line and since $5 > 3$, 5 is to the right of 3 on the number line.

Suppose we add two whole numbers and neither number is 0. We get a sum. The sum is greater than either of the 2 numbers that we added. The sum is a number to the right of either of the two numbers on the number line.

"2 is less than 5" because we can add the number 3 to 2 to get 5 as the sum. 2 is to the left of 5 on the number line. We write $2 < 5$ and read it "2 is less than 5."

"3 is less than 5" because we can add the number 2 to 3 to get 5 as the sum. 3 is to the left of 5 on the number line. We write $3 < 5$ and read it "3 is less than than 5".

Exercise Set 10

Copy each statement below. Write > or < in each blank so each mathematical sentence is true.

1. 5 (>) 6

2. 3 + 4 (>) 6

3. (11 [illegible]) (>) (11 − 5)

4. (30 + 20) (<) (31 + 21)

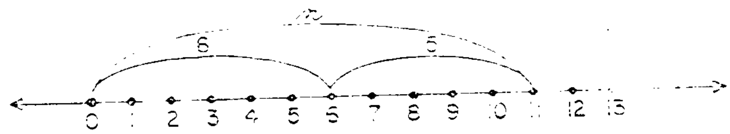
5. (53 + 40) (<) (53 + 41)

6. (200 + 600) (>) (200 + 700)

7. (1,200 + 1,000) (<) (1,200 + 2,000)

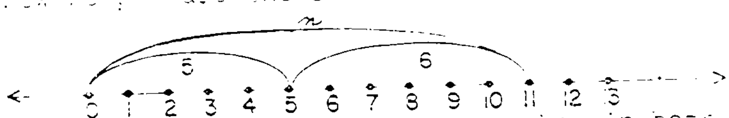
NUMBER LINE PICTURES AND MATHEMATICAL SENTENCES

Children now learn to use a number line to picture mathematical sentences. For example, 6 + 5 = n



Show pupils how the curved lines represent the sentence. The short curved lines represent the addends 6 and 5. The long curved line represents the sum.

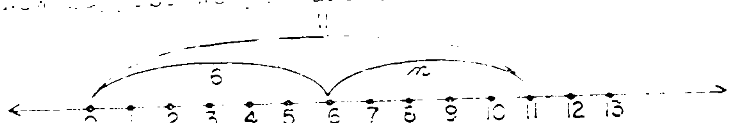
Now we picture the sentence 5 + 6 = n



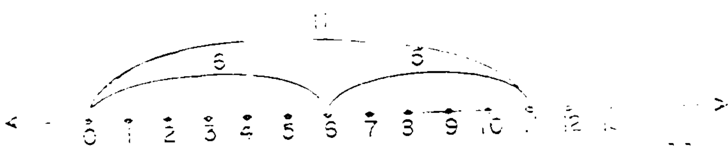
Since n represents the same number in both sums, we can use either picture to represent 6 + 5 = n and 5 + 6 = n. The number represented by n is 11.

You may need to consider several different examples and have the pupils make several drawings in order to get acceptance of one drawing to represent both 6 + 5 = 11 and 5 + 6 = 11. Notice the use here of the commutative property for addition.

Now suppose we picture the sentence 11 = 6 + n



The number represented by n is 5. Now the drawing

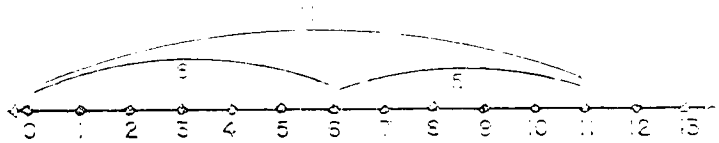


represents the four sentences

6 + 5 = 11
5 + 6 = 11
11 − 5 = 6
and 11 − 6 = 5

After this exploration, children should proceed to Exercise Set 11.

NUMBER LINE PICTURES AND MATHEMATICAL SENTENCES



A number line can be used to suggest mathematical sentences. The two short curves above the number line suggest the two addends. The long curve suggests the sum. The sentences suggested are

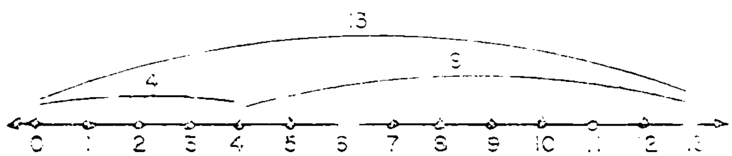
$$6 + 5 = 11$$

$$5 + 6 = 11$$

$$11 - 5 = 6$$

$$11 - 6 = 5$$

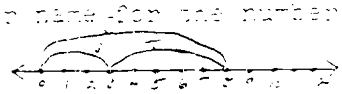
Exercise Set 11



1. Draw a number line as shown above. Draw curved lines to suggest this mathematical sentence $4 + 9 = n$.
   (a) What is the sum represented by n? 13,
   (b) Is it larger than 4? yes,
   (c) Is it larger than 9? yes,
   (d) When two whole numbers are added, is the sum always larger than either addend? no,
   (e) If your answer to (d) is, "No", give an example. $4 + 0 = 4$ When one addend is 0, the other addend is equal to the sum,

2. Draw a number line like the one in exercise 1. Use curves to suggest this subtraction: 8 − 3 = [illegible].

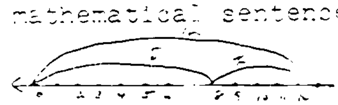
(a) What is another name for the number represented by [illegible] 5



(b) Is the unknown addend, represented by [illegible], larger than the sum? no

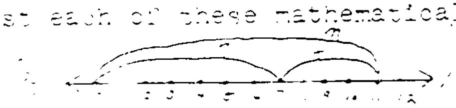
(c) Can an unknown addend ever be equal to the sum? yes

3. (a) How many units must be shown on a number line for you to picture this mathematical sentence. 12 − x = 8? 12

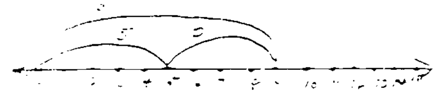


(b) Draw a number line. Use curves to suggest 12, the number represented by x, and 8 if 12 − x = 8.

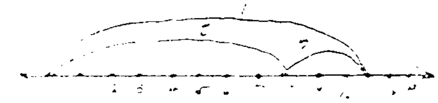
4. Draw number lines to suggest each of these mathematical sentences.



(a) [illegible]



(b) [illegible]



(c) [illegible]

BRAINTWISTERS

5. How many units must be shown on a number line to picture this mathematical sentence. 140 − s = 40. [illegible]

6. How many units must be shown on a number line to picture this mathematical sentence. [illegible] − 17 = 31 [illegible]

7. How many units must be shown on a number line to picture this mathematical sentence. n − 29 = 131 [illegible]

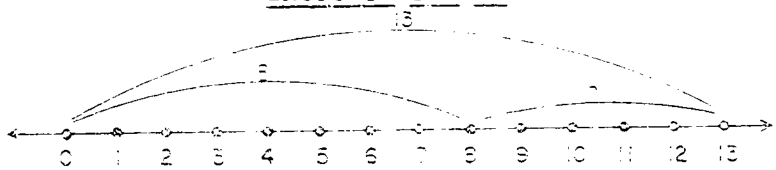
It is important that [illegible] be comprehensive. When pupils [illegible] answer "What number [illegible] so the mathematical sentence [illegible] = [illegible] is true?" They have a memorized response, [illegible] + [illegible] = [illegible]. They should know the [illegible] in order to answer "What number [illegible] so the mathematical sentence 62 + n = 130 [illegible] true." Here they do not have a memorized response. They should recognize 130 [illegible] the sum [illegible], [illegible] 62 as addends and that "n = 130 − 62." After writing n = 130 − 62 [illegible] use techniques of subtraction for finding n.

Use several sets of sentences such as [illegible] + n = [illegible], [illegible] = [illegible] − [illegible], [illegible] = [illegible], and [illegible] − n = [illegible]. All of these represent a relation between two addends [illegible] and n, and a sum [illegible]. In each case n = [illegible]. You may write a sentence such as [illegible] + n = 9 and ask for other ways to write it. (Answer n + [illegible] = [illegible], [illegible] − n = [illegible].) You may ask that mathematical sentences such as [illegible] + 2 = [illegible] or [illegible] = n + [illegible] be written using the symbol for subtraction. (n = [illegible] − [illegible], 2 = [illegible] − n) Similarly, ask pupils to write n = 10 − 4 or [illegible] using the symbol for addition. ([illegible] + 4 = 10, [illegible] = 10)

In Exercise [illegible] 11, exercises (a) and (b) [illegible] mathematical sentences. However, in exercise (a) there is no whole number which can be subtracted from 20 so the result is [illegible]0. In exercise (b) there is no [illegible] which can be added [illegible] 0.

MORE MATHEMATICAL SENTENCES

Exercise Set 11



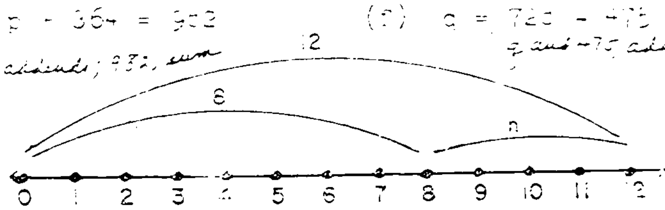
1. This number line suggests $n + 8 = 13$ or $8 + n = 13$. Does it also suggest $n = 13 - 8$? To answer "What number is the unknown addend n in $n + 8 = 13$?" can you think, "What number added to 8 is 13?" Can you also think, "n is 13 minus 8?" *Yes*

2. Tell the operation to use to answer, "What number is the unknown addend so each of these mathematical sentences is true?" *(Subtraction)* In each, tell which numbers are addends and which is the sum. Write your answers like this: (a) r and ? addends; 13, sum
   (r and ? addends; ?, sum)

   (a) n + [illegible] = 1[illegible] (d) $n = 456 - 194$ *(? and ? addends; 726, sum)*

   (b) r = 3,64[illegible] - 1,7[illegible] (e) $s - 524 = 1,726$ *(? and 153 addends; 3,649, sum)*

   (c) $p - 364 = 932$ (f) $q = 726 - 475$ *(q and 475 addends; 726, sum)* *(p and 364 addends; 932, sum)*

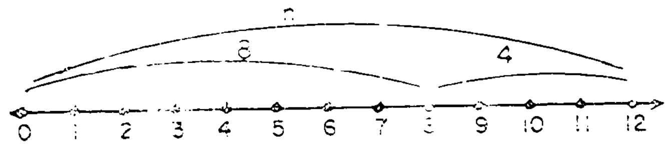


3. The above number line suggests $12 - n = 8$ or $12 - 8 = n$. Does it also suggest $8 + n = 12$? To answer, "What number is the unknown addend n in $12 - n = 8$?", can you think, "What number added to 8 is 12?" Can you also think, "n is 12 minus 8?" *Yes*

4. Tell the operation to use to answer, "What number is the unknown addend so each of these mathematical sentences is true?" Subtraction

(a) $15 - n = 11$ (c) $20 - n = 9$ (e) $13 - q = 8$

(b) $11 + n = 15$ (d) $p + 9 = 20$ (f) $8 + q = 13$



5. The above number line pictures $n = 8 + 4$. Does it also suggest $n - 4 = 8$? Also $n - 8 = 4$? Also $4 + 8 = n$? (yes)

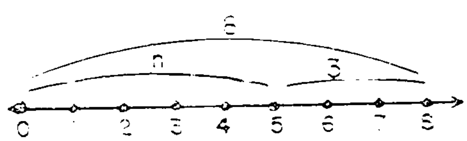
6. Tell the operation to use to answer, "What number is the sum so each of these mathematical sentences is true?" (Addition)

(a) $x - 6 = 10$ (c) $y - 15 = 25$ (e) $z - 7 = 8$

(b) $x = 10 + 6$ (d) $y = 15 + 25$ (f) $z = 7 + 8$

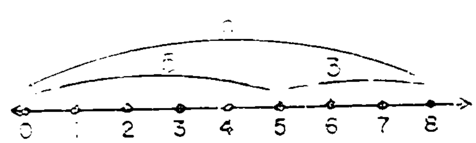
7. Write four mathematical sentences suggested by each picture.

(a)



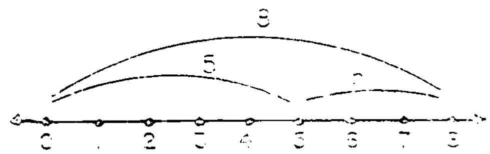
$n + 3 = 8$
$3 + n = 8$
$8 - 3 = n$
$8 - n = 3$

(b)



$5 + 3 = n$
$3 + 5 = n$
$n - 5 = 3$
$n - 3 = 5$

(c)



$n + 5 =$

$5 + n = 8$

$8 - 5 = n$

$8 - n = 5$

8. In each mathematical sentence tell which numbers are addends and which number is the sum. Then tell what operation you would use to find the number that p represents. [illegible] is done for you.

(a) $p = 11 - [illegible]$

p and [illegible] are addends. 11 is the sum. Subtraction.

(b) $p + 600 = 1,[illegible]$
p and 600 are the addends
[illegible] is the sum
Subtraction

(c) $67 + p = 136$
67 and p are the addends
[illegible] is the sum
subtraction

(d) $p - 75 = 113$
p and 75 are the addends
[illegible] is the sum
Addition

(e) $p - 39 = 206$
39 and 206 are the addends
p is the sum
Addition

(f) $p - 411 = 247$
[illegible] and 247 are the addends
p is the sum Addition

(g) $[illegible] - p = 542$
p and 542 are the addends
[illegible] is the sum
Subtraction

(h) $247 - p = 76$
p and 76 are the addends
247 is the sum
Subtraction

(i) $320 - p = 106$
p and 106 are the addends
320 is the sum
Subtraction

(j) $p - 40 = 630$
40 and 630 are the addends
p is the sum
Addition

9. What number does p represent so each mathematical sentence is true?

(a) $10 + p = 30$ [illegible] (c) $0 = p + 0$ [illegible] (e) $p = 15 - 5$ [illegible]

(b) $0 - p = 0$ [illegible] (d) $10 - p = 10$ [illegible] (f) $[illegible] - p = 15$ [illegible]

10. In which parts of exercise 9 is n not a counting number? [illegible]

BRAINTWISTERS:

11. (a) Write a mathematical sentence using n, 12, and 15. [illegible]

(b) Write another mathematical sentence using n, 12, and 15. n is a different number than in exercise (a). [illegible] − [illegible] = 5

12. Is there a whole number to replace n so each of these mathematical sentences is true? [illegible]

(a) 20 − n 30 (b) n + 30 = 20

[illegible] IN PROBLEM [illegible]

[illegible]

[illegible]

Teaching Procedure:

It is desirable [illegible] in this [illegible] solving some problems showing how [illegible] a mathematical sentence [illegible] in the problem between the [illegible] information" [illegible] the question that is to be answered.

The mathematical sentence [illegible] the numbers to be operated upon and the [illegible] of computation to be used in determining the number that has been represented by a letter in the mathematical sentence. Then finally an "answer sentence" is written to answer the question asked in the problem.

The following suggested procedure may help you in planning the exploration.

We now [illegible] mathematical [illegible] will help us in solving problems. Let us see if we can read a problem and then write a mathematical sentence [illegible] the information given in the problem.

Here is an easy one. A fourth grade class has 1[illegible] boys and 10 girls. How many children are in the class.

[illegible] we know about sets.

Set [illegible] 1[illegible] boys

Set of 10 girls

Now what is the question? [illegible] If we form the union of these two sets, how many children will there be in the union set?

15 boys 10 girls

We can write a mathematical sentence that fits this problem. With what numbers do we begin? (15 and 10, or 10 and 15).
Let us represent by n the number of objects in the union of the two sets. The number of objects in the union of two sets which have no common member is the sum of the numbers of objects in the two sets. So we write: $15 + 10 = n$ or $10 + 15 = n$.

$$25 = n$$

Let us summarize in this way. $15 + 10 = n$

$$\begin{array}{r} 15 \\ +10 \\ \hline 25 \end{array}$$

Answer sentence: There are 25 children in the class.

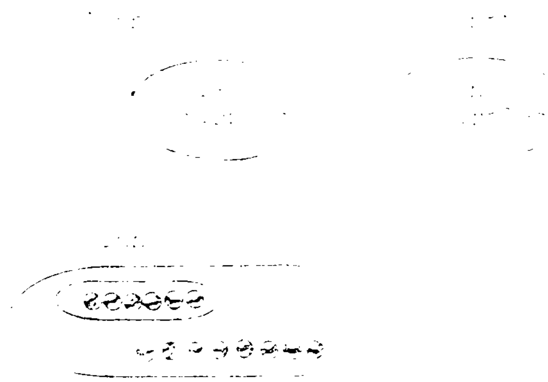
> The answer to the question asked in the problem may be obvious to all the children as soon as the question is asked. It is not intended that such a fairly long procedure as set down above shall be followed when the procedure to be used and the answer are obvious. But it is meant to be a guide for children who may not see the problem is an addition problem instead of a subtraction, or possibly a problem involving some other operation than either of these. Also, the larger part of the procedure is for assisting the pupils in writing a correct mathematical sentence. In general, finding the number represented in the mathematical sentence will be much easier than obtaining the mathematical sentence.

We must be careful in the way we think about addition. It is an operation on <u>numbers</u>. In this example we operated on the numbers 15 and 10. Addition is not an operation on children. We did not add 15 children and 10 children. We added the numbers 15 and 10. However, after we added the numbers, we may go back to our problem and say, "There are 25 children." This is called an "answer sentence."

[illegible]

[illegible]

[illegible]



A difficulty [illegible] drawing [illegible] to represent [illegible] partitioning of Jon's marbles into two sets containing [illegible] and n respectively is present since we do not know what number n represents. The pupil may "see" the sentence n + [illegible] = 14 or 14 − 6 = n more easily than he can picture the situation. If so then the picturization is a hindrance instead of help. [illegible] the above example with the pupils without picturization. Lead the pupils to the mathematical sentence and then the solution. Complete it by writing the "answer sentence."

We must be careful in the way we think about subtraction. It is an operation on numbers. In the example we have just used, we operate on the numbers 14 and 6. Subtraction is not an operation on marbles. [illegible] but we subtract numbers. However, after we subtract the numbers we may apply the result to marbles.

As the pupil meets more and more such [illegible] situations (like the one described [illegible]), he can learn to use the given bits of information to write mathematical sentences.

Let us look at some [illegible] situations and write mathematical sentences which describe them:

(a) [illegible] has a score of [illegible]. How many more points [illegible] a score of 1[illegible].

([illegible] + n = [illegible] or n = 1[illegible] − [illegible])

(b) A boy was given 10¢. He now has 2[illegible]¢. How much [illegible] (p + 10 = 2[illegible] or p = 2[illegible] − 10. Note that [illegible] symbol is not used.)

(c) In a class of [illegible] children, 12 went to the library. How many [illegible] room. (12 + s = [illegible] or s = 2[illegible] − 12.)

It should be noted that either of two sentences is correct for (a), (b), and (c) above. Pupils should not be forced to use one form in preference to the other. <u>They should write the mathematical sentence which is the result of their thinking</u>.

Ask the class for another example of the use of subtraction. These questions may serve as a guide in the discussion.

1. What question is asked in the problem?
2. What kinds of information are given in the problem?
3. What is the relationship between information and the question?
4. What mathematical sentence can we write to show this?
5. We use a letter in the mathematical sentence to represent a number. What operation do we use to find this number?
6. How can this number be used to answer the question asked in the problem?

Remember that the mathematical sentence a child writes results from the way he thinks of the question. [illegible] If a child writes 2 + n = 8, [illegible] two and what number is 8. If he writes n = 8 − 2, he may think eight minus two [illegible] number.

The teacher should use as many examples as are needed to help children further develop the idea of addition and subtraction as operations on numbers and as processes of finding the sum or the unknown addend. Many problems should be solved as follows: the numbers are separated from the situations, results are found, and the results are used to interpret the problems.

The teacher should have the pupil read and discuss the steps suggested on P [illegible]. The form shown at the bottom of P [illegible] <u>should be used by pupils in writing answers to the problems in Exercise S[illegible] [illegible].</u>

USING MATHEMATICAL SENTENCES IN PROBLEM SOLVING

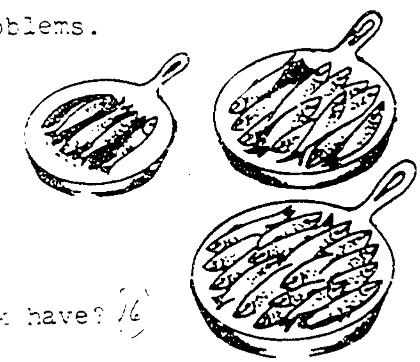
We have learned about the union of sets of objects.

We have learned about sets of objects within a set of objects.

How did the union of two sets help us understand addition?

How did sets within a set help us understand subtraction?

Now, let us see how we can use these ideas to help us answer questions in story problems.



1. Here is our first problem.

Dick caught 4 fish.

Jack caught 12 fish.

Dick gave his fish to Jack.

Then, how many fish does Jack have? (16)

(a) Is the set of fish that Jack now has, the union of two sets of fish? (yes)

(b) Does this mathematical sentence fit the problem:
$12 + 4 = n$? (yes) Why? (It shows that the set of 4 fish joined the set of 12 fish.)

Does this mathematical sentence fit the problem:
$4 + 12 = n$? (yes) Why? (Because $12 + 4 = 4 + 12$ by the commutative property of addition.)

(c) Now just think about the mathematical sentence:
$12 + 4 = n$ or $4 + 12 = n$ How do you find n?
(We add 12 and 4.)

(d) We find that $n = 16$.

We can answer the question in our problem.

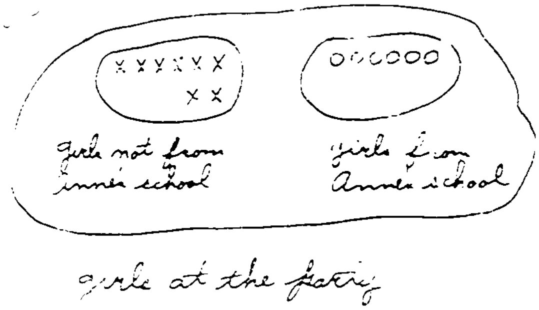
Jack now has 16 fish. (This sentence is called an answer sentence.)

2. Here is our second problem.

Anne had a birthday party: 14 girls came to the party. 6 of the girls were from Anne's school. How many were not from her school?

(a) Is the set of girls at Anne's party the union of two sets? (yes)

(b) Do we know the number of girls in one of the sets? (yes, there are 6)



(c) Can we represent the number of the girls in the other set by n? (yes)

(d) Does this mathematical sentence fit the problem: $14 - 6 = n$? or $n + 6 = 14$? (yes)

(e) What operation do we use to find n? (Subtraction)

We can now find $n = 8$.

The answer sentence is: There were 8 girls not from Anne's school

Summary

itical sentences are helpful in problem

sc hey help us to show number relationships

in orm. Here is a way you may use a

mat.em. sentence in solving a problem.

There are 22 children in a class. 10 of the chi ren are girls. How many are s?	Writing a mathematical sentence we could also write: $n = 22 - 10$, or $22 - 10 = n$, or $n + 10 = 22$
$10 + n = 22$ $\begin{array}{r} 22 \\ -10 \\ \hline 12 \end{array}$	Finding n
There are 12 boys in the class.	Answering the question by writing an answer sentence.

Exercise Set 13

Find the answer for each of the following problems. Use the order of working suggested on page F 1 .

1. Ann practiced the piano 35 minutes on Friday. She practiced 40 minutes on Saturday. How many minutes did she practice on the two days?
$35+40=n$ $\;\; 35 + 40 = 75$
Ann practiced 75 minutes on the two days

2. Mary read two books. One book had 42 pages. The other had 26 pages. How many pages did she read in these books.
$42+26=n$ $\;\; 42 + 26 = 68$
She read 68 pages in these books

3. Nancy has 9 crayons in her box. The box will hold 12. How many more crayons does she need to fill the box?
$9+n=12$, or $12-9=n$ $\;\; 12 - 9 = 3$
She needs 3 more crayons to fill the box

4. In a fish pond there are 25 black fish and 20 gold fish. How many fish are in the pond?
$25+20=n$ $\;\; 25 + 20 = 45$
There are 45 fish in the pond

5. Jim has a paper route. He has delivered 35 of his 49 papers. How many more papers does he have to deliver?
$35+n=49$ or $49-35=n$ $\;\; 49 - 35 = 14$
He has to deliver 14 more papers

6. There were 25 girls at a party. 15 of them were watching television. The others were playing. How many girls were playing?
$15+n=25$ or $25-15=n$ $\;\; 25 - 15 = 10$
Ten girls were playing

7. Jack has 59 stamps in two envelopes. In one envelope there are 24 stamps. How many stamps are there in the other envelope?
$24+n=59$ or $59-24=n$ $\;\; 59 - 24 = 35$
There are 35 stamps in the other envelope

8. Sue is saving to buy a book that costs 98 cents. She has 75 cents. How much more money does she have to save to buy the book?

$75 + n = 98$ or $98 - 75 = n$

$98 - 75 = 23$ She has to save 23 cents.

9. Tom spelled correctly 16 words on a test. He spelled 20 words correctly on another test. How many words did Tom spell correctly on both tests?

$16 + 20 = n$

$16 + 20 = 36$ Tom spelled 36 words correctly

10. Our auditorium was decorated with red balloons and white balloons. There were 63 balloons in all. If 41 balloons were red, how many were white?

$41 + n = 63$ or $63 - 41 = n$

$63 - 41 = 22$ There were 22 white balloons

11. At a popcorn sale, 29 bags were sold in one day. If 12 bags were sold in the morning, how many bags were sold in the afternoon?

$12 + n = 29$ or $29 - 12 = n$

$29 - 12 = 17$

17 bags were sold in the afternoon

12. David weighs 60 pounds. His little brother weighs 20 pounds. How many pounds do they weigh together?

$60 + 20 = n$

$60 + 20 = 80$ The two boys weighed 80 pounds

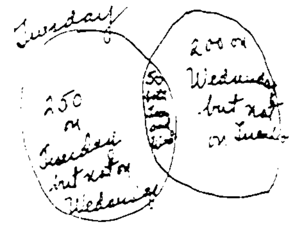
BRAINTWISTERS

13. There are 20 pupils in a class. The class has just two committees to plan a party. The refreshments committee has 7 members. The games committee has 5 members. James, Mary, and Bob are on both committees. How many pupils are on just one committee? (6) How many pupils are on either one or two committees? (9) How many pupils in the class are not on any committee? (11)

14. 300 pupils attended the school football game on Tuesday. 250 pupils attended the school football game on Wednesday. 50 pupils attended both the games.

How many pupils attended just one of the games? (450)

How many pupils attended the game on Wednesday that did not attend the game on Tuesday? (200



DOING AND UNDOING-ADDITION AND SUBTRACTION

Objective: To help children understand that addition of a number and subtraction of that same number will undo each other

Mathematicians use a more sophisticated language for this. They say addition of a number and subtraction of the same number are <u>inverse operations</u>. It is probably better not to use the term "inverse" with children. However, the idea is an important one.

Vocabulary: Doing, undoing

Teaching Procedures:

It will be helpful to introduce the idea of inverse operations through the use of "doing and undoing" activities which already are very familiar to children.

Begin by writing several pairs of sentences such as these on the chalkboard, using names of children in your class:

> Sue walked forward 5 steps.
> Sue walked backward 5 steps.
>
> Joe opened the window.
> Joe closed the window.

For each pair of sentences, ask whether or not the second action "undoes" the first action.

Have the children give pairs of sentences of their own in which doing and undoing actions are involved.

Now write several single sentences on the chalkboard such as these:

> Jane earned 25 cents.
> Bill got on his bicycle.

For each sentence, have children suggest an "undoing" sentence to go with it, e.g., Jane spent 24 cents; Bill got off his bicycle.

You may use pages P 105 and P 106 of the pupil's book to develop further the ideas of doing and undoing. The exercises should be discussed and completed with all children together. However, each child should prepare and complete his own copy of the chart in exercise 1. After working through pages P 105 and P 106 assign Exercise Sets 14 and 15.

If the children in your class are able to understand the form of symbolism used below you may use this way to summarize the idea of addition and subtraction as "undoing" operations:

If a and b represent the whole numbers

(1) $(a - b) + b = a$

(2) $(a + b) - b = a$

Note on Braintwister, exercise 10, page P 106: Have pupils look up the meaning of the word identity in the dictionary. They should choose the appropriate meaning. One they might find is: "the state of being exactly the same." In addition, 0 is called the identity element because if 0 is one addend, the sum is the same number as the other addend.

DOING AND UNDOING - ADDITION AND SUBTRACTION

There are many actions that undo other actions. For example, Jack found a dime. He lost that dime.

1. Complete the chart below with the missing "doing" or "undoing" actions.

Doing	Undoing
1. stand	sit
2. open	close
3. dress	undress
4. tie	untie
5. button	unbutton

2. Give some sentences like this one which tell about both doing and undoing:

"Ellen opened her book and then closed it."

We have seen that one action may undo another action. This exercise will help us to see if subtracting a number will undo adding that same number.

3. (a) Think of 8. Add 3 and then subtract 3.
What is your result? 8
Finish this sentence: (8 + 3) - 3 = 8

(b) Think of 10. Subtract 6 and then add 6.
What is your result? 10
Finish this sentence: (10 - 6) + 6 = 10.

(c) What must you do to [illegible] to get [illegible] *Subtract* —

(d) What must you do to 6 – [illegible] to get [illegible] *Add* —

<u>Adding a number and subtracting that same number undo each other</u>. For example, if we start with 9, then add 2, and then subtract 2, the result is 9, the number we started with. Subtracting 2 undid adding 2. We can write

$$(9 + 2) - 2 = 9.$$

<u>Subtracting a number and adding that same number undo each other</u>. For example, if we start with 5, then subtract 3, and then add 3, the result is 5, the number we started with. Adding 3 undid subtracting 3. We can write

$$(5 - 3) + 3 = 5.$$

Exercise Set 14

Answer Yes or No for exercises 1 - 3

1. Is the result the same number for (a) and (b) below? yes

   (a) Start with 5, add 2, and then subtract 2.

   (b) Start with 5 and add 0.

   (c) Is $(5 + 2) - 2 = 5 + 0$? yes

2. Is the result the same number for (a) and (b) below? yes

   (a) Start with 14, subtract 6, and then add 6.

   (b) Start with 14 and add 0.

   (c) Is $(14 - 6) + 6 = 14 + 0$? yes

3. Is the result the same number for (a) and (b) below? yes

   (a) Start with n, add 6, and then subtract 6.

   (b) Start with n, and add 0.

   (c) Is $(n + 6) - 6 = n + 0$? yes

4. Write mathematical sentences for (a) and (b).

   (a) Start with 9, add 5, and then subtract 5.

   (b) Start with 9, and add zero.

   (c) Are the results the same for (a) and (b)? yes

   $(9 + 5) - 5 = 9$

   $9 + 0 = 9$

5. Write mathematical sentences for (a) and (b).

   (a) Start with n, add [illegible], and then subtract [illegible]. $n + [illegible] - [illegible] = n$

   (b) Start with n, and add zero. $n + 0 = n$

   (c) Are the results the same for (a) and (b)? yes

6. Is it true that if you start with any whole number and add 0, the result is that whole number? yes

   Give some examples. $[illegible] + 0 = [illegible]$ $[illegible] + 0 = [illegible]$ [illegible]

7. Apply the "undoing" idea to these operations.

   Exercises (a) and (b) are done for you.

DO	UNDO
(a) [illegible] + 2	[illegible] − 2 = [illegible]
(b) [illegible] − [illegible]	[illegible] + [illegible] = [illegible]
(c) 5 + 3	8 − 3 = 5
(d) 15 − 10	[illegible] + 10 = [illegible]
(e) 25 + 20	45 − 20 = 25
(f) [illegible] + n	[illegible] − n = [illegible]
(g) [illegible] − [illegible]	[illegible] + [illegible] = [illegible]
(h) [illegible] − n	[illegible] + n = [illegible]

8. What operation is used [illegible] find [illegible] in each of these true mathematical sentences.

   (a) [illegible] = n   [illegible]   (f) [illegible] − 30 = n   subtraction

   (b) n = [illegible]   subtraction   (g) [illegible] = n − 51   subtraction

   (c) [illegible] = n [illegible]   subtraction   (h) [illegible] − [illegible] = n   addition

   (d) n − [illegible]   subtraction   (i) [illegible] − 100 [illegible]   subtraction

   (e) n = 52 [illegible]   addition

9. What number is n for each of the exercises (a) to (i) in exercise 8?

(a) $n = 11$ (d) $n = 41$ (g) $n = 44$
(b) $n = 3$ (e) $n = 97$ (h) $n = 164$
(c) $n = 1$ (f) $n = 31$ (i) $n = 26$

10. BRAINTWISTER: The mathematician calls 0 the identity element for addition. What is the meaning of identity?

(Identity means one and the same)

What do you think identity element for additon means?

(If zero is added to any number, the result is that same number)

11. BRAINTWISTER: (a) Two numbers to be operated on are 8 and (10 - 10). The operation is addition. What is the result? (8)

(b) In exercise (a) if (10 - 10) is replaced by 0, what is the result? (8)

(c) Two whole numbers to be operated on are a and (6 - 6). The operation is addition. What is the result? a

(d) Two whole numbers to be operated on are m and (n - n). The operation is addition. What is the result? m

12. BRAINTWISTER: In exercise 11, replace the word addition by the word subtraction. Answer each of the four parts of exercise 11.

(a) 8
(b) 8
(c) a
(d) m

Exercise Set 15

Write a mathematical sentence for each of these problems. Then solve.

1. (a) Tom and Peter had 72 cookies for their class picnic. The boys ate 12 on their way to the picnic. How many were left for the picnic? ($72 = n + 12$, or $72 - 12 = n$; $n = 60$; 60 cookies were left)

   (b) Peter's mother had to bring one dozen more cookies for the class picnic. How many cookies were there for the class? ($60 + 12 = n$; $n = 72$; There were 72 cookies for the class)

   (c) Show how the operation of exercise (b) undoes the operation of exercise (a) by use of a mathematical sentence. ($(72 - 12) + 12 = 72$)

2. (a) Margaret has 12 addition exercises to do. Her teacher gave her 8 more exercises.

   (b) Margaret does 8 exercises.

   (c) Show how the operation part (b) undoes the operation of part (a) by use of a mathematical sentence. ($(12 + 8) - 8 = 12$)

3. (a) Jon had \$45 in the bank. He spent \$20 during the summer for swimming lessons.

   (b) Jon earned \$ 20 and put it in the bank.

   (c) Show how the operation of part (b) undoes the operation of part (a) by use of a mathematical sentence. ($(45 - 20) + 20 = 45$)

MORE ABOUT ADDITION AND SUBTRACTION OF WHOLE NUMBERS

Objective: To help children understand that the operation of addition is always possible within the set of whole numbers, but the operation of subtraction is not always possible within the set of whole numbers.

Vocabulary: Closure, closed

Teaching Procedure:

If we, like fourth-grade children, know and use only whole numbers in operations, we can find the sum of any pair of whole numbers. However, we cannot subtract any pair of whole numbers, e.g., (2 - 5). This illustrates that the set of whole numbers is closed under addition and not closed under subtraction.

The mathematician in describing the property of closure would say, "The set of whole numbers is closed under the operation of addition," to express the fact that when two whole numbers are added, there always is a sum within the set of whole numbers. In further describing the closure property he would say, "The set of whole numbers is not closed under subtraction," to express the fact that when two whole numbers are subtracted, there may or may not be a result within the set of whole numbers.

"Closure" is probably not a term to be used with the children, but the property is an important one to the understanding of the operations of addition and subtraction. To children it means, "I can always find a whole number which is the sum when I add a pair of whole numbers. But it is not always possible to find a whole number for an unknown addend when I know the sum and one addend." For example, if the sum is 11 and one addend is 12, there is no whole number that can be the unknown addend.

To reinforce the idea that some things are possible and some impossible under certain conditions, exercises 1 and 2, P111 should be studied and discussed. In finding answers to exercises 3 and 4 on page P 111, the following development may be used to begin this work.

The numbers we use now in our class are the set of all the whole numbers: 0, 1, 2, 3, and so on. Do you think there is a whole number to use as a sum for any pair of whole numbers we might add?

To help us answer this question, let us operate on some pairs of whole numbers, and put our results on the chalkboard. (Use chart as below). Is there always a whole number which is the sum for each of the pairs of whole numbers we tried? Do you think you could find a pair of whole numbers whose sum would not be a whole number?

Addition Pairs	Is there a whole number for the sum?
57, 93	Yes
2, 9	Yes
0, 6	Yes
99, 100	Yes
100, 99	Yes
0, 0	Yes

How many pairs for which there is no whole number sum would you need to discover before you could say that we cannot always add within the set of whole numbers. (One) Do you think we always can add within the set of whole numbers? (Yes). We describe this by saying that the set of whole numbers is <u>closed</u> under addition. This just means that the sum of two whole numbers is a whole number.

Now follow a similar procedure using pairs of whole numbers and the operation of subtraction, as in the chart at the right. Emphasize that order is important. When we see 9, 5 we write 9 - 5 = n and we think "What number added to 5 will give 9?"

Subtraction Pairs	Is there a whole number which is the unknown addend?
9, 5	Yes
5, 9	No
256, 40	Yes
40, 256	No
0, 3	No
3, 0	Yes
6, 6	Yes

Here children should be led to see that for several pairs of whole numbers it was not possible to subtract because there was no whole number to use as the unknown addend. So, we cannot always subtract within the set of whole numbers. Children should know that this means the set of whole numbers is not closed under subtraction.

Children might make the following generalizations: When only the set of whole numbers is used, we can always add but we cannot always subtract. We can subtract only when the sum is greater than or equal to the missing addend. This is summarized on page P 111.

The following development should precede the assignment of Exercise Sets 1[illegible] and 1[illegible].

Let's work with [illegible] (1, 2, 3, 4, 5) and pretend we know no other numbers.

Can we add any pair of these numbers and get a sum which is in the set? Try some. (2 + 2 = 4; 3 + 2 = 5; [illegible]; [illegible]) There is no number in the set for the sum of 3 and 4 and the sum of 3 and 3.) Can you find a sum in the set for some pairs? (Yes, but there is no sum in this set for other pairs.) Let us try each pair including adding each number to itself. Is it true, then, if we know only the set of numbers 1, 2, 3, 4, and 5, we can find a sum in the set for each pair of addends?

Do you suppose we can subtract any pair of the numbers? Try some pairs. ([illegible] - 2 = 3; 4 - 3 = 1; 3 - 3 = 0. There is no number in the set to express the unknown addend for 3 - 3 = 0.) Can you subtract any number in this set from any other number in this set and get a number in the set?

Is it true, then, that if we know only the set of numbers 1, 2, 3, 4, and 5 we cannot always find an unknown addend in an addition sentence?

Notice. If we knew only 1, 2, 3, 4, and 5, it would not always be possible to find a number in the set as a result in addition or in subtraction. So we can say the set {1, 2, 3, 4, 5} is not closed under addition or subtraction.

In the addition operation on the numbers in {1, 2, 3, 4, 5} we can always get a sum of any two of the numbers. But the sum is not always in the set {1, 2, 3, 4, 5}. Since the sum is not always in the set, then the set is not closed under (or with respect to) addition.

> It is not always possible to subtract any two numbers in the set {1, 2, 3, 4, 5}. For example the pupils cannot now find n in 2 - 5 = n since they know <u>no</u> number that can be added to 5 to get 2 as the sum. Certainly then the number for n is not in {1, 2, 3, 4, 5} and the set is not closed under subtraction.
>
> Exercise Sets 16 and 17 may be used now. The teacher should carefully choose the exercises which are to be used for the ability groups in his class.

MORE ABOUT ADDITION AND SUBTRACTION OF WHOLE NUMBERS

1. Think of the set of the first 5 whole numbers.

   A = {0, 1, 2, 3, 4}

   (a) Add any 2 of these numbers. You are permitted to add one of these numbers to itself. What numbers do you get for sums? (0, 1, 2, 3, 4, 5, 6, 7, 8)

   (b) Is each of the sums in Set A? No

   (c) Which sums are in Set A? (0, 1, 2, 3, 4)

   (d) Why are the other sums not in Set A? (Because the sums are 5, 6, 7 and 8 and these numbers are not members of Set A.)

2. Suppose you have just the numbers in Set S.

   S = {8, 9, 10, 11}.

   Find the sum of any two numbers in Set S? Are any of these sums members of Set S? No

   Is Set S closed under addition? No

3. If two whole numbers are added, is the result always a whole number? Yes Try some examples. (4 + 5 = 9, 0 + 12 = 12 etc.)

4. If two whole numbers are subtracted, is the result always a whole number? No Try some examples. Do you sometimes get a whole number? Yes 8 − 5 = 3

   (5 − 8 = There is no whole number for this subtraction)

It is always possible to add two whole numbers because there is always a whole number to use as a sum. This means the set of whole numbers is <u>closed</u> under addition.

It is not always possible to subtract two whole numbers because there is not always a whole number to use as the other addend. There is no whole number n so that $3 - 5 = n$. This means that the set of whole numbers is <u>not</u> <u>closed</u> under subtraction.

Exercise Set 16

1. Pretend you know only the set of even whole numbers. Write a few of the members of the set. $\{2, 4, 6, 8, 10, 12, 14 \cdots\}$

   (a) Choose six pairs from the set. Add them using a form such as shown at the right. (Remember that you can use only the even whole numbers as addends.)

26	even
14	even
40	even

   (b) What can you say about the sum for each addition you tried? (All of the sums are even)

   (c) When you add two even whole numbers, do you expect to get a sum which is always odd? (no) Always even? (yes) Sometimes odd and sometimes even? (no)

   (d) How many pairs of numbers did you try? Try enough pairs so that you are sure of your answer in (c).

   (e) Is the sum of any two even whole numbers a number in the set of even whole numbers? (yes) Is the set of even whole numbers closed under addition? (yes)

2. BRAINTWISTER: Set A = {0, 3, 6, 9, 12, 15, 18, 21, ...} Think of all pairs of Set A such as 0 and 3, 3 and 3, 6 and 9, and so on. Think of the sum of each pair. Call this set of sums Set B. Write Set B. Is every member of Set B a member of Set A? (yes)
(Set B = {0, 3, 6, 9, 12, 15, 18, 21 ...})

3. Think of the set of odd whole numbers. Write a few members of the set. ({1, 3, 5, 7, 9, 11, 13, 15 ...})

   (a) Choose six pairs from the set. Add them. Use a form such as shown at the right. (Remember that you can use only the odd whole numbers as addends.)

11	odd
19	odd
30	even

   (b) You have only odd numbers in this set. Is there a number to use for a sum in each pair you choose? (yes) Is the sum an odd whole number? (no)

   (c) When you add a pair of odd numbers, do you expect to get a result which is always odd? (no) Always even? (yes) Sometimes odd and sometimes even? (no)

   (d) How many pairs of numbers did you try? Try enough pairs so that you are sure of your answer in (c).

   (e) Is the sum of any two odd whole numbers a number within the set of odd whole numbers? (no) Is the set of all odd whole numbers closed under addition? (no)

4. BRAINTWISTER: Set A = {1, 4, 7, 10, 13, 16, 19, 22, ...} Think of some pairs of Set A such as 1 and 4, 4 and 10, 7 and 7, and so on. Think of the sum of each pair. Call this set of sums Set B. Write Set B. Is any member of Set B a member of Set A? (no) Is Set A closed under addition? (no) Set B = {2, 5, 8, 11, 14, 17 ...}

Exercise Set 17

BRAINTWISTER SET

1. Pretend you know only the set of numbers {1, 2, 3, 4}. This means that in this exercise you may use only the numbers 1, 2, 3, 4.

   (a) Copy and fill in, wherever possible, the addition chart at the right.

+	1	2	3	4
1	(2)	(3)	(4)	
2	(3)	(4)		
3	(4)			
4				

   (b) Did you fill in each space of the chart? (No)

   (c) If not, why not? (Some of the answers are greater than 4 or this set is not closed under addition)

2. Pretend you know only the set of numbers {6, 8, 10, 12, 14}.

   (a) Copy and fill in, wherever possible, the addition chart at the right.

+	6	8	10	12	14
6	(12)	(14)			
8	(14)				
10					
12					
14					

   (b) Did you fill in each space of the chart? No

   (c) If not, why not?

   Some of the answers are greater than 14 or the set is not closed under addition

3. Pretend you know only the set of numbers {1, 3, 5, 7}.

   (a) Copy and fill in, wherever possible, the addition chart at the right.

+	1	3	5	7
1				
3				
5				
7				

   (b) Did you fill in any space of the chart? (No)

   (c) If not, why not? (None of the sums are odd numbers)

4. (a) In exercise 1 is the set closed under addition? (No)

   (b) In exercise 2 is the set closed under addition? (No)

   (c) In exercise 3 is the set closed under addition? (No)

MORE PROBLEM SOLVING

Objective: To help children further their problem solving ability through the use of mathematical sentences in situations requiring addition and subtraction of numbers whose numerals have many digits

To children, problem solving should be a process for finding the answer to the question asked in a problem. To find the answer, the relationship in the problem must first be identified

Vocabulary: Relationship

Teaching Procedure:

The relationship in a problem as used here is a description of the essential way the elements in a problem are related. The elements may be numbers, geometric ideas, or other concepts. The description may use words or mathematical symbols. In the problems which follow, the description of the relationship in mathematical symbols is a mathematical sentence.

Problem solving should be developed with emphasis on the relationship in a problem. The relationship in the problem is identified and written as a mathematical sentence. The sentence is studied to determine a method of finding an answer to the question asked in the problem. An operation is performed and the result used to answer the question asked. The answer to the problem is called "answer sentence."

The children should think of the relationship in the problem at all stages in the problem solving process. It is the key to problem solving.

To learn more about solving problems, we shall study some problems to find the relationship among the numbers in the problem. When we write a mathematical sentence, we are describing how the numbers are related.

You may wish to select one or two problems from Exercise Set 1 and follow the suggested procedures below, or you may wish to use other problems and have children do problems in Exercise Set 1 on their own. In either case here are some questions to guide the discussion.

Using the first problem in Exercise Set 18, ask

1. What question is to be answered?
   (a) How far did the Smiths travel on the round-trip?

2. What are the bits of information in the problem?
   (a) The Smiths traveled 2140 miles while going.
   (b) The Smiths traveled 2037 miles while returning.

3. <u>What mathematical sentence can we write to express the relationship</u> between the question and the bits of information?

   $2,140 + 2,037 = n$

   Do not expect all children to write the same mathematical sentence.

Following this, children may use the computational form to determine the result.

$$\begin{array}{r} 2,140 \\ +2,037 \\ \hline 4,177 \end{array}$$

They are then able to answer the question asked in the problem.

The Smiths traveled 4,177 miles.

In a similar way, the class might solve the following problem:

Margaret has read 113 pages. Her book has 247 pages. How many more pages has she left to read?

1. What question is to be answered?
   How many more pages does she have to read?

2. What are the bits of information in the problem?
   She has read 113 pages of a book.
   The book has 247 pages.

3. What mathematical sentence can we write to express the relationship between the question and the bits of information?

   $113 + n = 247$ or $\;\; = 247 - 113$

MORE PROBLEM SOLVING

Solving written problems is an exercise in careful thinking. You may need to read a problem several times. Be sure you understand what question is asked in the problem.

Look for the statements in the problem that give you information. This information may be in more than one statement.

Write a mathematical sentence to show the relationships in the problem. (You may wish to look at pages 99 - 101.)

Study the mathematical sentence and decide what operation to use. Then carry out the operation.

Write an answer sentence to explain that your answer from the operation is an answer to the question in the problem.

<u>Example</u>.

Jack has [illegible] stamps in his collection. Bill has 325 stamps in his collection. Jack has how many more stamps than Bill?

What mathematical sentence shows the relationship in the problem?

[illegible]35 − 32[illegible] or 32[illegible] + n = [illegible]

What operation do we use to find n? Subtraction

[illegible] n = 112

The answer sentence is "Jack has 112 more stamps than Bill."

Exercise Set 11

1. During the summer the Smith family traveled 2,140 miles on their way to Yellowstone Park. They traveled 2,037 miles on their way home. How many miles did they travel altogether?

   2,140 + 2,037 = n

   The Smith family traveled [illegible] miles

2. Margaret has read 11[illegible] pages. Her book has 247 pages. How many more pages has she left to read?

   247 = [illegible] + n, [illegible] 247 − [illegible] = n; 132. Margaret had 13[illegible] more pages to read.

3. The Stone family car has traveled 12,547 miles. The Brown family car has traveled 11,325 miles. How much farther has the Stone family car traveled than the Brown family car?

   [illegible] 325 = n, 12,547 − 11,325 = n

   $$\begin{array}{r} 12,547 \\ -\ 11,325 \\ \hline 1,222 \end{array}$$

   The Stone family has traveled 1,222 miles farther.

4. How much larger is three thousand, two hundred seventy-five than two thousand, one hundred thirty-three?
$3275 - 2133 = n$ 3275 is 1142 larger than 2133 $2133 + n = 3275$ $\begin{array}{r}3275\\-2133\\\hline 1142\end{array}$

5. Mary and her mother collect old buttons. Mary's mother has 275. Mary has just begun her collection and has 124. How many buttons do Mary and her mother have in all?
$275 + 124 = n$
$\begin{array}{r}275\\+124\\\hline 399\end{array}$ Mary and her mother have 399 buttons

6. The Grant School has 225 Red Cross boxes. They filled 114 of theirs. How many more boxes are to be filled?
$225 = 114 + n$ or $225 - 114 = n$ 111 more boxes need to be filled $\begin{array}{r}225\\-114\\\hline 111\end{array}$

7. On Saturday, 1,462 people bought tickets for the Little League World Series games. On Sunday, 2,526 people bought tickets for the game. How many tickets were bought for the games on these two days?
$1462 + 2526 = n$
3,988 people bought tickets for the game

8. Helen was born in 1960. How old will she be on her birthday this year? Answers will vary, depending upon the year

THE ASSOCIATIVE PROPERTY OF ADDITION

It should be emphasized that addition is a binary operation, namely just two numbers can be added in one addition operation. Consequently in adding three numbers, the addition operation must be performed twice. This may be done by grouping, or associating the numbers to be added. For example, to find 9 + 5 + 11 we may find (9 + 5) + 11 or 9 + (5 + 11). The children will recognize the need for the associative property as examples like these are used:

14 + 27 + 36

137 + 12 + 18

2136 + 164 + 295

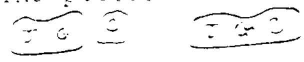
This may help them see that just two numbers can be added at the same time; also that they might simplify the addition by one grouping in preference to the other. In the examples:

	(14 + 36) + 27 = 50 + 27
but	14 + (36 + 27) = 14 + 63
	137 + (12 + 18) = 137 + 30
but	(137 + 12) + 18 = 149 + 18
and	(2136 + 164) + 295 = 2300 + 295
but	2136 + (164 + 295) = 2136 + 459

In each the first grouping is obviously the easier one.

This work can be introduced by the union of three sets. No two of the sets have a member in common.

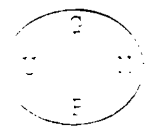
Activity	Sets to be Combined	Does grouping of things affect the results?
Meeting	5 members of the Jones family; 4 members of the Gump family; [illegible] members of the Clarky family	No. If the Gump family joins the Jones family and then they are joined by the Clarky family, the same people will be present as if the Jones family joins the Clarky family and they then are joined by the Gump family.

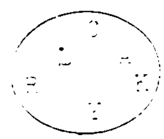
In this example let the students picture the union of the Jones and Gump families and then the union of this union set with the Clancy family. The picture might be something like this. 

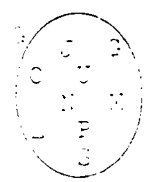
Another way to picture the union of these three disjoint sets (the meeting of the three families) would be to designate the members of each family by letters. For example, the five members of the Jones family by J, O, N, E, S; the four members of the Gump family by G, U, M, P, and the six members of the Clancy family by C, L, A, R, K, Y.

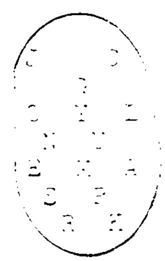
Then:

J 

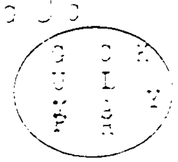
G 

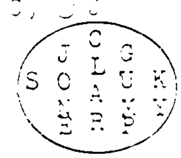
C 

J ∪ G 

(J ∪ G) ∪ C 

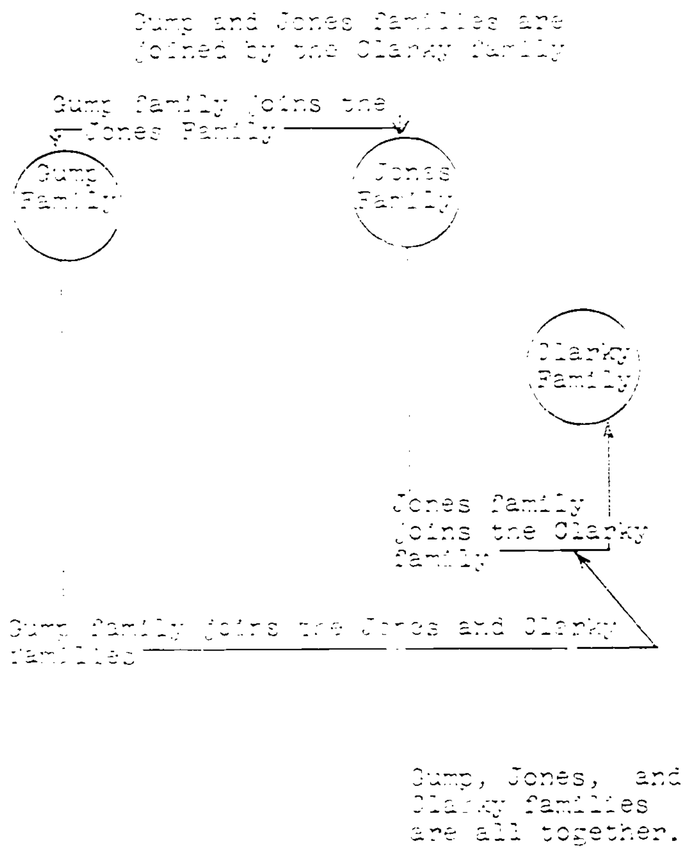
Follow this diagram to picture J ∪ (G ∪ C)

G ∪ C 

(G ∪ C) ∪ J 

and [illegible]

By families:



Pages P 119, P 120 and P 121 in the Pupil's Book can now be discussed and pupils can work Exercise Set 19 independently. A discussion of this Exercise Set would be fruitful after pupils have completed it.

THE ASSOCIATIVE PROPERTY FOR ADDITION

1. Suppose you were told to add 6, 7, and 5.

   (a) Can you add three numbers at the same time? No, we can only add 2 numbers at a time. Addition is taking of just two numbers and getting a third number. The third number is called the sum.

   (b) We can add 6 and 7 because we can add just two numbers. Let's write $(6 + 7) + 5$. This means we will add 6 and 7. We get 13. Then we add 13 and 5 and get 18. We have the sum of the three numbers 6, 7, 5. How many numbers did we add at any one time? (two)

   (c) We can also write $6 + (7 + 5)$. This means we will add 7 and 5. We get 12. Then we add 6 and 12.

   (d) Are the final sums in (b) and (c) the same? Yes, (18)

It is not possible to add more than two numbers at a time. If we have more than two numbers to add, we must group just two numbers. For example, if we want to add 536, 451, and 612, we cannot do all three of them in one operation. We can add 536 and 451 and then add 612 to this sum. Or we can add 451 and 612, and then add this sum to 536. We could write $(536 + 451) + 612$ or $536 + (451 + 612)$ to show how we add the three numbers.

In the example at the top of this page we must add just two numbers at a time. We must group just two numbers together. To do this for $6 + 7 + 5$ we can write

$$(6 + 7) + 5 = n$$

The parentheses means that we are grouping the 6 and 7 and we think of 6 + 7 as one number, 13. Then the sum is 13 + 5 or 18. We could write.

$$6 + (7 + 5).$$

This means we are grouping the 7 and 5 and we think of this as one number, 12. Then the sum is 6 + 12 or 18. (6 + 7) + 5 and 6 + (7 + 5) are each names for the same number, 18. The way in which we grouped the numbers did not change the sum. When we group 6 + 7 + 5 as (6 + 7) + 5 or as 6 + (7 + 5), we are using the <u>associative property</u> for addition. We must group the numbers by twos since we can add <u>just two</u> numbers at a time.

2. If we use the associative property for addition to write 3 + 2 + 4 = n, we would write:

$$(3 + 2) + 4 = n$$
$$5 + 4 = n$$
$$9 = n$$

or

$$3 + (2 + 4) = n$$
$$3 + 6 = n$$
$$9 = n$$

Find each sum. Use the associative property for addition as was done above.

(a) 2 + 1 + 5 = n

(b) 6 + 3 + 2 = n

(c) 8 + 2 + 3 = n

a) 2+1+5=n or 2+1+5=n
(2+1)+5=n   2+(1+5)=n
3+5=n   2+6=n
8=n   8=n
b) 6+3+2=n   6+3+2=n
(6+3)+2=n or 6+(3+2)=n
9+2=n   6+5=n
11=n   11=n

8+2+3=n
(8+2)+3=n
10+3=n
13=n
or
8+2+3=n
8+(2+3)=n
8+5=n
13=n

3\. (a) Tell how to do this operation: $(6 + 7) + 5$.
(Add 6 and 7 first. Then add 13 and 5.)

(b) Tell how to do this operation: $6 + (7 + 5)$.
(Add 7 and 5 first. Then add 6 and 12.)

(c) Why must we group two of the numbers in adding 6, 7, 5? (Because addition is an operation on just two numbers.)

4\. (a) What is the result of $(6 + 7) + 5$? (18)

(b) What is the result of $6 + (7 + 5)$? (18)

(c) Is $(6 + 7) + 5 = 6 + (7 + 5)$? (Yes)

5\. (a) Is $(3 + 4) + 5 = 3 + (4 + 5)$? (Yes)

(b) Are $(3 + 4) + 5$ and $3 + (4 + 5)$ different names for the same number? (Yes)

(c) In what way is $(3 + 4) + 5$ different from $3 + (4 + 5)$? (The addends are grouped differently.)

Summary

Adding of three numbers must be done in two steps. You may add 63, 24, and 82 in either of two ways if the order is not changed.

$$(63 + 24) + 82 = 87 + 82 = 169$$

$$63 + (24 + 82) = 63 + 106 = 169$$

The sum is the same even if we did group the addends differently.

So, we can write

$$(63 - 24) + 82 = 63 + (24 + 82).$$

Exercise Set 1

1. The associative property is used in finding the sum of 14 and 5. You have not called it by [illegible] name but you have used it:

$$14 + 5 = (10 + 4) + 5 \qquad \text{Step I}$$
$$= 10 + (4 + 5) \qquad \text{Step II}$$
$$= 19 \qquad \text{Step III}$$

Is $(10 + 4) + 5 = 10 + (4 + 5)$ illustrative of the associative property for addition? Yes

(b) Is 19 a different name for [illegible] + 5)?

2. The associative property can help you do some additions easier. These are ways of adding $15 + 9 + 11$.

(a)
$$15 + 9 + 11 = (15 + 9) + 11$$
$$= 24 + 11$$
$$= 35$$

(b)
$$15 + 9 + 11 = 15 + (9 + 11)$$
$$= 15 + 20$$
$$= 35$$

Is the sum the same using either method (a) or (b)? Yes

Many pupils like method (b) better when they add without paper and pencil. Why? The results for 9 + 11 and 15 + 20 are easier to do mentally than the results for 15 + 9 and 24 + 11.

3. What property of addition is illustrated by [illegible]? It is the associative property for addition.

4. What properties of addition are illustrated by $(3 + [illegible]) + 5 = 3 + (5 + 7)$? Associative property and commutative property

3 + 5 + 7 = 3 + (5 + 7)

3 + 5 + 7 = 3 + (7 + 5)

5. (a) Does ([illegible] - 3) - 2 = [illegible] - (3 - 2)? No

(b) Does ([illegible] - [illegible]) - [illegible] = [illegible] - ([illegible] - [illegible])? No

(c) Is subtraction an associative operation? No

6. To find the sum [illegible] + 2 + [illegible] a boy wrote 4 + 2 = [illegible] + 7=13. The statement he wrote is wrong. Why? 4 + 2 is not equal to 6 + 7

For exercises 7 - 11 write the correct words, numerals and mathematical sentences to complete this chart. Remember that numerals in parentheses name one number.

	Numbers Operated On	Result	Operation Used	Mathematical Sentence
7.	87, 56	31	Subtraction	87 − 56 = 31
8.	(8 + 6), 5	19	Addition	(8 + 6) + 5 = 19
9.	11, (9 + 6)	26	Addition	11 + (9 + 6) = 26
10.	(6 + 5), (8 + 4)	23	Addition	(6 + 5) + (8 + 4) = 23
11.	27, (8 + 8)	11	Subtraction	27 − (8 + 8) = 11

12. Write + or − in each blank so each sentence is true.

(a) (12 + 8) − 7 = 13

(b) 13 − (8 − 6) = 15

(c) (42 + 61) + 52 = 155

(d) 54 − (38 + 11) = 5

13. Write parentheses to help you do these.

(a) 4 + 6 + 19

(b) 11 + 9 + 25

(c) 17 + 3 + 29

(d) 58 + 12 + 8

(a) (4 + 6) + 19

(b) (11 + 9) + 25

(c) (17 + 3) + 29

(d) 58 + (12 + 8)

REVIEW

Objective: To help pupils [illegible] summarize the major concepts of the chapter and practice the important skills

Teaching Procedures:

The teacher should decide which method of using the material is most appropriate for his class. If pupils work the exercises independently, a class discussion of various ways of solving the exercises would be valuable. Another approach would be a discussion of a few parts of each exercise with pupils working the remainder independently. Exercise Set 20 is a set designed to help pupils improve their problem solving ability by writing questions that require the operation of addition or subtraction on the numbers in the problem. Here pupils have an opportunity to write the language which calls for an operation.

Exercise 16, page P 12[illegible]. There are five counting numbers (20, 21, 22, 23, and 24) between 19 and 25. If $n = 25 - 19$, then $n = 6$.

REVIEW

[illegible]

Some of the properties that you have studied are reviewed below. Tell if the statements are always true, sometimes true, or never true. Give examples.

Property Stated for Addition

1. (a) If 0 is added to a whole number, the result is that whole number. Always true

2. (a) If two whole numbers are added, the result is a whole number. Always true

3. (a) If the order of adding two whole numbers is changed, the sum is unchanged. Always true

Property Stated for Subtraction

1. (b) If 0 is subtracted from a whole number, the result is that whole number. (Always true)

2. (b) If two whole numbers are subtracted, the result is a whole number. (Sometimes true)

3. (b) If the order of subtracting two whole numbers is changed, the unknown addend is unchanged. (Some times true)

4. Find n so each mathematical sentence is true.

(a) $6 + n = 6 + 9$ (9

(b) $12 + n = 1$[illegible] $+ 12$ 7

(c) $n + 21 = 21 + 42$ 42

(d) ([illegible] + [illegible]) + n = [illegible] + ([illegible] + 4) (5

(e) (12 + [illegible]) + n = 12 + ([illegible] + 5) (3)

(f) (3 + [illegible]) + $(n + 2)$ = (5 + 2) + (3 + [illegible]) (5)

P-2

5. Place parentheses in 18 - 12 - 2 so the result is [illegible].

Is (18 - 12) - 2 = [illegible] (18 - 12 - 2) = [illegible]

6. Place parentheses so

a. 24 - 12 - [illegible] = [illegible] [illegible] - 12 - [illegible] = [illegible]

b. [illegible] - [illegible] - 20 - [illegible] d. [illegible] - [illegible] - [illegible] = [illegible]

7. On your paper [illegible] the letter [illegible] beside each true mathematical sentence. [illegible]

a. 17 - 19 = 19 - 17 b. [illegible] - [illegible] = [illegible] - 17

[illegible]

[illegible]

[illegible]

[illegible]

8. Is [illegible] - [illegible] = [illegible] - [illegible]? Without adding [illegible] the [illegible] Do you see that [illegible] and [illegible] are the two addends. Tell which [illegible] you used.

[illegible] *associative property for addition*

[illegible] (8 + 12) + [illegible] = 19

commutative property for addition

[illegible] *commutative property for addition*

[illegible] *commutative property for addition*

[illegible] *associative property for addition*

9. (a) Does $(2 + 7) - 2 = 2 + (7 - 2)$? Yes

(b) Does $2 + (7 - 2) = 7 + (2 - 2)$? Yes

(c) Does $(2 + 7) - 2 = 7 + (2 - 2)$? Yes

10. Copy the letters (a) to (d) on your paper.

Beside each letter, write the correct words, numerals, and mathematical sentences needed to complete this chart.

	Numbers Operated On	Result	Operation Used	Mathematical Sentence
(a)	(7 + 4), 8	3	Subtraction	$(7+4) - 8 = 3$
(b)	20, (40 + 30)	90	Addition	$20 + (40+30) = 90$
(c)	(3[illegible] − 30), (40 − 40)	0	Subtraction	$(30-30) - (40-40) = 0$
(d)	(63 − 61), (199 − 1)	222	Addition	$(63-61) + (199+1) = 222$

11. Writing parentheses helps you do the addition, $23 + 19 + 11$ without paper and pencil. See the box on the right. Write parentheses to help you do these.

$23 + 19 + 11 = n$
$23 + (19 + 11) = n$
$23 + 30 = n$
$53 = n$

(a) $12 + 8 + 16$ (d) $16 + 17 + 26$

(b) $87 + 3 + 42$ (e) $15 + 36 + 14$

(c) $18 + 82 + 75$ (f) [illegible]

12. Find what number n represents so that each mathematical sentence is true. Be careful. There may be no answer, one answer, or even more than one answer.

(a) $(3 + 2) + 5 = n$ $n = 10$ (f) $3 + n = n$ no answer

(b) $(3 + 2) + n = 8$ $n = 3$ (g) $n = n$ any number

(c) $(3 + 2) - n = 8$ no answer (h) $1 + n = n + 1$ any number

(d) $1 + n = n$ no answer (i) $7 + n = n + 7$ any number

(e) $2 + n = n$ no answer

13. Does $(5 - 4) - 1 = 5 - (4 - 1)$? Why? yes because they name the same number

14. Does $(6 - 4) - 1 = 6 - (4 - 1)$? Why? no because $\neq 3$

15. BRAINTWISTER: Make up two examples like exercise 13 and exercise 14. Which one is true and which one is false? answers will vary

16. BRAINTWISTER: How many counting numbers are greater than 19 and less than 25? Is $n = 25 - 19$ the correct relationship for this problem? no

Exercise Set 21

Write a question that requires addition for each of exercises 1 to 3. Sample answers given

1. Tom and Bob were collecting old clothes for the church drive. Bob worked for 45 minutes on Monday and 30 minutes on Tuesday. How long did Bob work on the two days?

2. Tom called at 12 houses and Bob called at 11 houses. Did Tom & Bob called at how many houses in all?

3. Bob collected 15 coats and 21 dresses.
How many pieces of clothing did Bob collect?

4. Write a question that requires subtraction for each of the exercises 1 to 3.

For each of the exercises 5 through 7 write: (a) the numbers operated on; (b) the mathematical sentence; (c) the result. Arrange your work in a chart like this.

	Numbers Operated On	Mathematical Sentence	Result
5.	12, 8	$8 + n = 12$ or $12 - 8 = n$	4
6.	37, 23	$23 + n = 37$ or $n = 37 - 23$	(14)
7.	28, 5	$5 + n = 28$ $n = 28 - 5$	(23)

5. How much less than a dozen is 8?

6. Bob had a collection of 37 toy airplanes. One day he could find only 23. How many were gone.

7. Five of the 28 girls on the playground went back early into the school building. How many remained on the playground

Exercise Set 22

1. Study this: $(6 - n) - 1 = 6 - (n + 1)$

   (a) What is the largest whole number n can be? (5)

   (b) What is the smallest whole number n can be? (0)

   (c) Find all the whole numbers n can be. (0, 1, 2, 3, 4, 5)

2. BRAINTWISTER: Tell what whole number n is so that each mathematical sentence is true. Be careful. There may be no answer, one answer, or more than one answer.

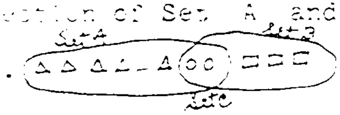
   (a) $n - 1 = 1 - n$ (1) (d) $n + 50 = 50 + n$ (any whole number)

   (b) $n - 10 = 10 - n$ (10) (e) $n = n - 1$ (No number)

   (c) $6 + n = n + 6$ any whole number (f) $10 - n = n$ (5)

Exercise Set 23

BRAINTWISTER SET

Pretend you have two sets, A and B. Set A has 8 members. Set B has 5 members. The intersection of Set A and Set B is Set C. Set C has 2 members. 

1. Make a drawing of these intersecting sets.

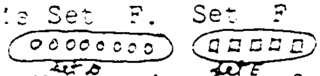
2. How many members in Set A are not in Set B? (6)

3. How many members are in Set B and not in Set A? (3)

4. How many members are in the union of Set A and Set B? (11)

5. Write the mathematical sentence from which you got the answer to exercise 4. ($8 - 5 - 2 =$ or $6 + 3 + 2 = 11$)

Pretend you have two sets. D and E. Set D has 8 members. Set E has 5 members. The intersection of Set D and Set E is Set F. Set F is an empty set.



6. Make a drawing of these intersecting sets.

7. How many members in Set D are not in Set E? (8)

8. How many members in Set E are not in D? (5)

9. How many members are in the union of Set D and E? (13)

10. Write the mathematical sentence from which you got the answer to exercise 9. (8+5=13)

11. SUPER BRAINTWISTER: At Grant School there is a Mathematics Club. The members of the club are also members of certain sets. The members of Set A have read the magazine, Popular Mathematics. The members of Set B have read the book, Mathematics is Fun. The members of Set C have read both the book and the magazine. Set A has 6 members, Set B has 5 members, and Set C has 3 members.

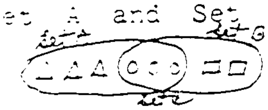
    (a) Are all members of Set C also members of Set A? (Yes)

    (b) Are all members of Set C also members of Set B? (Yes)

    (c) How many members of Set C are also members of Set A? (3)

    (d) How many members of Set C are also members of Set B? (3)

    (e) The intersection of Set A and Set B has how many members? (3)

    (f) Make a drawing of the intersecting sets.

    (g) How many members are there in the club? (8)

Chapter 4

PROPERTIES OF MULTIPLICATION AND DIVISION

PURPOSE OF UNIT

The purpose of this unit is to help the children understand the nature and properties of multiplication and division as operations of mathematics. It is assumed that the children have had little experience with multiplication and division. They need to consider the nature of these operations carefully in order to understand later the actual techniques for multiplying and dividing numbers whose numerals have many digits. They will need also immediate knowledge of the basic multiplication facts. It is the intent of this unit to develop this knowledge.

Multiplication usually is taught as a special case of addition when the addends are equal. Here it is taught primarily as a mathematical operation with certain characteristic properties. The habit of viewing multiplication as an operation in its own right should be of great value to children. This will be useful later when they study the multiplication of rational numbers where reduction to addition is less direct.

The emphasis of the unit will be on multiplication as an operation. The operation of division is defined in terms of multiplication. It is an operation on two numbers, a product and a factor, determining an unknown factor. This approach emphasizes the role of multiplication facts. If children know the multiplication facts, they need no independent division facts.

The properties to be developed are: the commutative property of multiplication, properties of 0 and 1, the closure property, the distributive property of multiplication over addition, and the associative property of multiplication.

No property of division is explicitly studied. The distributive property of division over addition is developed now because of its usefulness later in studying the "long division" algorism.

MATHEMATICAL BACKGROUND

In the mathematical background at the beginning of Chapter 3 of the fourth grade, the concept of a mathematical operation was defined and discussed. The principal concern of this unit is the operation of multiplication. Division is described as the operation of finding an unknown factor in a multiplication situation. In a mathematical sentence of the form $a \times b = c$, a and b are called factors of c and c is their product. If a and b are given, they are multiplied to obtain c. If c and a are given, then c divided by a is b. This operation is denoted by sentences of the form $b = c \div a$.

In this connection, it is suggested that the teacher avoid use of the term "quotient". This term is not needed and may tend to divorce division from multiplication. Division should not be indicated by a bar, e.g. $b = \frac{c}{a}$, until the relation of division to rational numbers is introduced in subsequent study.

Throughout the unit, analogies and contrasts among multiplication, subtraction, and division are stressed. For example, addition and multiplication are commutative and associative operations. The set of whole numbers is not closed under subtraction or division.

Just as subtracting a number is the inverse of adding the same number, so dividing by a number (not 0) is the inverse of multiplying by the same number. Although the term "inverse" is not used with children, they can understand that division by a number will "undo" multiplication by that number, and multiplication by a number will "undo" division by that number. For example, if we start with 6, perform the multiplication 6×8, we obtain the product 48. If, then, we divide 48 by 8, we come back to 6. $(6 \times 8) \div 8 = 6$. Similarly $(12 \div 3) \times 3 = 4 \times 3 = 12$. It is in this sense, as doing and undoing, that we expect children to understand the inverse relationship between multiplying and dividing.

Zero and one play special roles in multiplication and division. The characteristic property of 0 is that, for every whole number a, $0 \times a = a \times 0 = 0$. Also, $1 \times a = a \times 1 = a$ is peculiar to the number 1.

The situation for division is more complicated. If c denotes any whole number, then $c \div 1 = c$. If c is any counting number, then $0 \div c = 0$, but $c \div 0$ is meaningless. The expression $0 \div 0$ is ambiguous. These properties are derived from the definition of $c \div a = b$ as another way of writing $c = a \times b$.

Because the product of every pair of whole numbers is a whole number, we say that the set of whole numbers is *closed* under multiplication. The set of whole numbers is *not* closed under division. Children will not use this language, but can understand that in the set of whole numbers, although the operation of multiplication is always possible, the operation of division is not always possible.

The point should be made repeatedly that in this unit, *we operate on whole numbers to obtain another whole number*. $13 \div 6$ has no meaning as an operation on whole numbers. The fact that the set of fractional numbers is closed under the operation of division is irrelevant in the context of this unit.

Reversing the order of factors $(7 \times 9$ or $9 \times 7)$ does not alter the product. This property is called the commutative property of multiplication. It effectively reduces the number of multiplication facts to be remembered and is useful in simplifying calculations.

The distributive property gives the relation between multiplication and addition. It is the most general statement of the role of addition in finding products. It is best explained by examples and a general symbolic formula. For example, 7×13 can be written as $7 \times (10 + 3)$. The relation

$$7 \times (10 + 3) = (7 \times 10) + (7 \times 3)$$

exemplifies the distributive property and shows how it can be used to reduce a problem to known multiplication facts and addition. The distributive property of multiplication over

addition may be stated generally as:

$$a \times (b + c) = (a \times b) + (a \times c).$$

Because multiplication is a commutative operation, we may write also

$$(b + c) \times a = (b \times a) + (c \times a).$$

There is a corresponding distributive property of division given by the formula $(b + c) \div a = (b \div a) + (c \div a)$. As an example:

$$(9 + 6) \div 3 = (9 \div 3) + (6 \div 3).$$

This is true since $15 \div 3 = 3 + 2$.

Some caution must be observed in giving examples of this property. If we are operating within the set of rational numbers, $(8 + 7) \div 3 = (8 \div 3) + (7 \div 3)$ is meaningful and true, because the sentence states that $\frac{15}{3} = \frac{8}{3} + \frac{7}{3}$. It has no meaning in the context of the set of whole numbers, because the divisions $8 \div 3$ and $7 \div 3$ cannot be performed using only whole numbers.

There are two ways to multiply 7, 9, and 2 keeping the numbers in that order. One is shown by $(7 \times 9) \times 2$. The other is $7 \times (9 \times 2)$. The fact that they give the same result ($63 \times 2 = 126$ and $7 \times 18 = 126$) is an instance of the associative property of multiplication. The general statement is given by

$$(a \times b) \times c = a \times (b \times c).$$

Consequently, we can interpret $a \times b \times c$ in either of these ways without affecting the product.

Teaching the Unit

This chapter is organized in the following manner.

1. There are suggested explorations which appear only in the teachers' commentary.
2. There are explorations and working together sections which appear in the pupil text.
3. There are pupil exercises to be done independently.

It is recommended that the teacher follow the explorations in the teachers' commentary preceding the explorations or working together with pupils on sections in the pupil text. The latter are designed to be read and discussed <u>with</u> pupils. Pupil explorations and working together sections serve as summaries, in many cases, and offer pupils a record of the development of ideas throughout the unit. It is not intended that all children do all exercises. Conversely, some exercise sets may require supplementing.

Arrays are used throughout the work on multiplication as a physical model with which this operation is associated. The use of arrays does not imply that pupils are not to learn the multiplication facts for immediate recall. Arrays serve merely as aids to understanding. Facility with the multiplication facts is essential before division and more difficult multiplication are attempted.

ARRAYS

Objective: To help children understand how arrays may be used in describing
1. the matching of two sets of objects
2. the arrangement of objects

Materials: Round gummed labels for making demonstration arrays, graph paper with $\frac{1}{4}$ inch squares for pupils to construct arrays, straightedge, and crayons

Vocabulary: array, row, column, elements

Exploration: Sue has three new blouses, a white one, a yellow one, and a pink one. She has two new skirts, a red one and a blue one.

How many times can Sue come to school wearing a different matching of blouse and skirt? (She can wear a white blouse and a red skirt one day. She can wear a white blouse and a blue skirt another day, etc. She can come to school 6 times wearing a different matching of blouse and skirt.)

> The children may need to see the matchings written out before they are put into a chart. One way to write them is:
>
> RW, RY, RP
> BW, BY, BP
>
> Use of flannel pieces on a felt board may be helpful, too.

Let's make a chart:

	White Blouse	Yellow Blouse	Pink Blouse
Red Skirt			
Blue Skirt			

Sue can wear a red skirt and a white blouse one time. I will put a dot in the chart under White Blouse and to the right of Red Skirt. Describe another matching she can wear, and put a dot on the chart. (Sue can wear a blue skirt and a white blouse, etc.)

The discussion should continue until the chart is completed.

The number of matchings represented by all the dots in the chart can be determined by counting or by adding. Some children may use what they already know about multiplication to see quickly that the total number of dots is the <u>product</u> of the number in each row and the number in each column. It should be useful in every class, however, to analyze the chart in terms of counting and addition. There are several ways to determine the number of dots in any chart. One is illustrated by:

. 1	. 3	. 5
. 2	. 4	. 6

another by:

. 1	. 2	. 3
. 4	. 5	. 6

The elements in each of the charts (arrays) used in the classwork suggested in the remainder of this exploration section can be counted in various ways. When the charts are put on the board or made by the children, numerals should often be written next to the dots during this initial contact. This practice should be useful a little later when counting is replaced by addition. If some children count the arrays by twos, threes, etc., at the beginning, let them do so. The central point of the classwork is to reach observations like the following: (1) There are 6 matchings of 2 skirts with 3 blouses. (2) This means that a chart (array) with 2 rows and 3 columns consists of 6 elements. Note that in conventional practice, the rows are named first and the columns second. In this array

.	.	.
.	.	.

there are 2 rows and 3 columns. This is a 2 by 3 array.

Continue asking questions until understanding is evident. Such questions as the following may be asked.

1. How many different matchings could Sue make of the red skirt and the 3 blouses?
2. How many different matchings could Sue make of the blue skirt and the 3 blouses?
3. How many different matchings could Sue make of the 2 skirts and the 3 blouses?
4. How is each matching to be represented on the chart?

If Sue's mother bought her a green blouse, she would have 2 skirts and 4 blouses. Now how many times can she wear a different matching of skirt and blouse to school? Extend the chart we made to include the green blouse.

	White Blouse	Yellow Blouse	Pink Blouse	Green Blouse
Red Skirt	.	.	.	.
Blue Skirt	.		.	.

The same type of questioning used with the previous array should be used with this array.

Make a chart of dots to show the matching of 2 skirts and 5 blouses. Make a dot for each matching.

Use former line of questioning.

Look at all of the charts we have made. I will put them on the board in an orderly fashion beginning with the one we had for Sue's 2 skirts and 3 blouses, then the one for Sue's 2 skirts and 4 blouses, etc. I will show with a numeral the number of rows and columns of dots.

From [illegible] first chart, what question can you answer? (How many tim[illegible] can [illegible] with a different matching of [illegible])

[illegible] chart in a similar [illegible]

[illegible] [illegible] a new [illegible]

[illegible] [illegible] [illegible] can answer all of the questions [illegible] only the last chart. Using it, we can answer all the [illegible] can use only the part of the chart which [illegible], I can erase all the charts except the [illegible]. Do you agree? [illegible] and 16.

In this [illegible] skirts and blouses. [illegible] you still answer questions about the number of times she can [illegible] different matchings of skirt and blouse? [illegible] blouses. She can come to [illegible] matchings of skirt and blouse.

Chapter 4

PROPERTIES OF MULTIPLICATION AND DIVISION

ARRAYS

Exploration

John went to the ice cream shop to buy cake and ice cream. The shop sells 3 kinds of cake, chocolate, angel food, and coconut; and 4 flavors of ice cream, vanilla, chocolate, strawberry, and cherry. How many choices of one kind of cake and one kind of ice cream does he have? Copy the following headings on the chalkboard and make a chart of his choices.

	Vanilla Ice Cream	Chocolate Ice Cream	Strawberry Ice Cream	Cherry Ice Cream
Chocolate Cake	•	•	•	•
Angel Food Cake	•	•	•	•
Coconut Cake	•	•	•	•

Use the chart to answer these questions.

How many choices does he have with 1 kind of cake and 4 flavors of ice cream? (4)

How many choices does he have with 2 kinds of cake and 4 flavors of ice cream? (8)

How many choices does he have with 3 kinds of cake and 4 kinds of ice cream? (12)

Could we make a chart of his choices without using the words cake and ice cream and letting dots stand for the choices? (Yes) Would the chart look like this? (Yes)

•	•	•	•
•	•	•	•
•	•	•	•

If peppermint ice cream were added to his choices of flavors of ice cream, how many choices of three kinds of cake and five flavors of ice cream does he have? Make a chart to answer the question.

```
.  .  .  .  .
.  .  .  .  .
.  .  .  .  .
```

Make other charts to show that John has [illegible] choices of ice cream flavors, [illegible] choices of ice cream flavors, [illegible] choices of ice cream flavors, [illegible] choices of ice cream flavors.

From these charts what questions can you answer? (About his choices.) How are the charts alike? (They are alike because they have 3 rows of dots. They are different because each chart has a different number of columns.) How are the charts different? Can you use one chart to answer all the questions about John's choices? (Yes we need only the last chart. We can use only the part of the chart which is needed.)

Did your chart look like this?

```
.   .   .   .   .   .   .   .   .

.   .   .   .   .   .   .   .   .

.   .   .   .   .   .   .   .   .
```

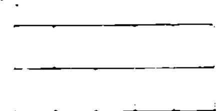
This chart is called an array. It is an orderly arrangement of objects in rows and columns. In this array there are 3 rows and 9 columns. There are 27 elements in the array.

What arrays are shown below? (Array (a) is a 4 by 6 array.) (Array (b) is a 2 by 5 array.)

(a)

```
.  .  .  .  .  .
.  .  .  .  .  .
.  .  .  .  .  .
.  .  .  .  .  .
```

(b)



How many rows are in array (a)? (4 rows)

How many columns are in array (a)? (6 columns)

How many rows are in array (b)? (2 rows)

How many columns are in array (b)? (5 columns)

How many elements are in each array? (Array (a) has 24 elements. Array (b) has 10 elements.)

We have seen that an array can be used to show all matchings of one set with another. Arrays are useful in other situations. Some sets of objects are themselves arranged in arrays. Here are some examples:

a) eggs in a carton

b) panes of glass in a window or door

c) seats in an auditorium

d) pieces of candy in a box

e) crayons in a box

f) linoleum blocks on our room floor, etc.

Can you think of some sets of objects arranged in arrays to add to the list of examples?

Draw an array of dots which represents a possible arrangement of [illegible] panes in a window. How many panes are there in a window whose panes form a 3 by 4 array? (12)

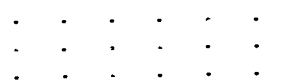
> The teacher should be prepared to answer questions relating to the two different uses of arrays in representing situations. Each dot in an array representing matchings of ice cream and cake depicts neither ice cream nor cake. However, in the examples immediately above, the objects under discussion themselves form an array, and the dots or other marks used in symbolizing such an array actually represent those objects.

Exercise Set 1

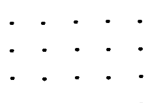
1. Copy the following table. Use the arrays shown below to help you complete the table.

Exercise	Rows	Columns
Example:	3	7
a.	4	5
b.	3	4
c.	4	2
d.	4	1
e.	6	6
f.	5	8

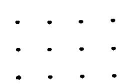
Example:



a.



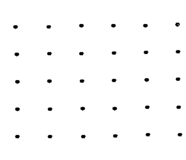
b.



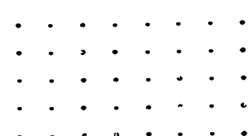
c.

d.

e.



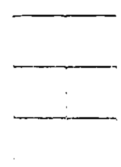
f.

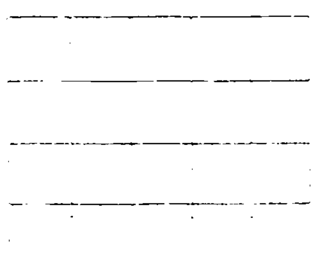


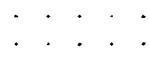
2. Copy and fill in the table below.

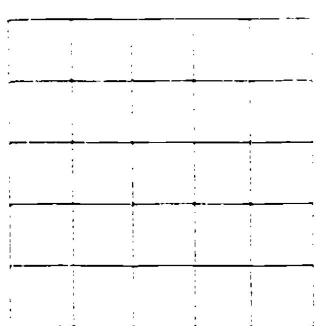
Exercise	Rows	Columns	Elements
Example:	2	5	10
a.	3	2	6
b	[illegible]	5	20
c.	3	5	15
d	5	5	25
e	7	4	28

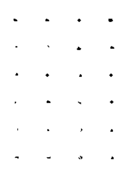
Example: 

a. 

b. 

c. 

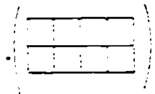
d. 

e. 

2?

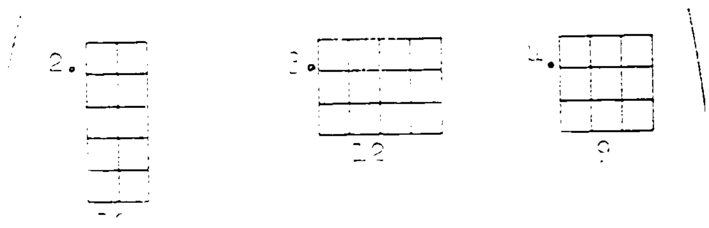
MAKING ARRAYS

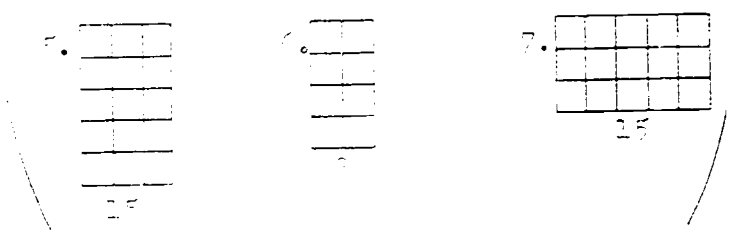
<u>Exercise Set 2</u>

1. Draw an array that has 2 rows and 4 columns. 

For each of the following pairs of numbers, draw an array. The first number tells the number of rows. The second number tells the number of columns. Under each array, write the number of elements in the array.

2. 5 , 2
3. 3 , 4
4. 3 , 3
5. 5 , 3
6. 4 , 2
7. 3 , 5

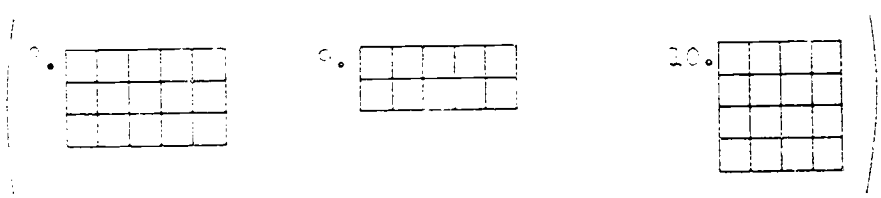




8. Draw an array of 15 elements that has 3 rows.

9. Draw an array that has 10 elements and 5 columns.

10. Draw an array that has 16 elements and 4 rows.



MULTIPLICATION

Objective: To help children understand that multiplication is an operation on two numbers giving a third number

Materials: Felt board and cut outs to use with the chart on page 251

Vocabulary: Multiplication, operation

Exploration: Must we count the dots in a 2 by 5, or 2 by 6, or 2 by 7 array to find the number of matchings of a set of 2 elements with other sets? Is there a shorter way? A 2 by 5 array can be separated into five 2 by 1 arrays.

```
. . . . .
. . . . .
```

It has, therefore, 2 + 2 + 2 + 2 + 2 or 10 elements in it. A 2 by 6 array can be separated into a 2 by 5 array and a 2 by 1 array.

```
| . . . . . |   | . |
| . . . . . |   | . |
```

From this we see that it has 10 + 2 or 12 elements in it. Does anyone see a shorter way to use addition in finding the number of elements in a 2 by 6 array? (Split it into two 1 by 6 arrays. This shows that there are 6 + 6 or 12 elements.)

Let us talk about two arrays. One of them has 2 rows and 9 columns. One of them has 3 rows and 9 columns. What use can we make of them? (We can use the 2 by 9 array to find the number of matchings of a set of 2 elements and any set of 9 elements or less. We can use the 3 by 9 array to find the number of matchings in an array composed of a set of 3 elements with any set of 9 elements or less.)

```
. . . . . . . . .        . . . . . . . . .
. . . . . . . . .        . . . . . . . . .
                         . . . . . . . . .
```

Array A   Array B

I will put two other arrays on the chalkboard.

Array C   Array D

What do you notice about Array A and Array C? (They have the same number of elements. Array A has 2 rows of 4 columns. Array C has 4 rows of 2 columns.) Can you use Array A to show the arrangement of 8 chairs in a room? (Yes, they could form a 2 by 4 array.) Could you use Array C? (Yes, we could think of 4 rows with 2 chairs each. It really makes no difference. We need an array with 2 elements on one side and 4 elements on another.) Can they be used to find the answer to other questions? (Yes)

What do you notice about Array B and Array D? (They can be used to answer the same questions. Each has 3 elements on one side and 9 elements on the other.)

Look at the two arrays and answer these questions:

a) How many matchings can you make of a set of 3 with a set of 4? (12)

b) How many matchings can you make of a set of 5 with a set of 2? (10)

c) How many dots are there in an array if the two sets matching have 5 elements and 3 elements? (15)

d) How many dots are there in an array if the two sets matched have 3 elements and 6 elements? (18)

In these questions, I told you the number of elements in each of two sets. What did you tell me? (The number of elements in an array showing the matchings of the members of the 2 sets.)

Here is a chart which gives you the number of elements in each of two sets. Write the number of elements in the array which shows the matchings of the two sets.

Number of Elements in Each of Two Sets	Number of Matchings in The Array
2 , 2	4
2 , 3	6
2 , 4	8
2 , 5	10
.	.
.	.
.	.
6 , 2	12
7 , 2	14
8 , 2	16
9 , 2	18

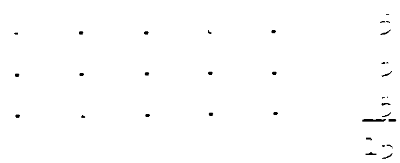
In making this chart given 2 numbers, you can get a third number. Given 6 , 2 , you get 12. How can you think of 3 and 5 to get 15? We have a name which describes "Thinking about two numbers and getting a third number." What is it? (When we think of two numbers and get a third number, we are operating on the two numbers.)

When we begin with the number of elements in one set and the number of elements in another and think of the matching, we associate the number of matchings with the operation called multiplication. These matchings can be arranged in an orderly fashion where the number of rows corresponds to the number of elements in one set, and the number of columns corresponds to the number of elements in the other set.

Since most classes will have been introduced to multiplication as an operation derived from addition, it will be necessary to identify the two descriptions of the operation:

a) think of 3 and 5 and get the sum 5 + 5 + 5, and

b) think of 3 and 5 and get the number of elements in a 3 by 5 array. Discussion of several diagrams of the following sort, based on situations already used, should be helpful.

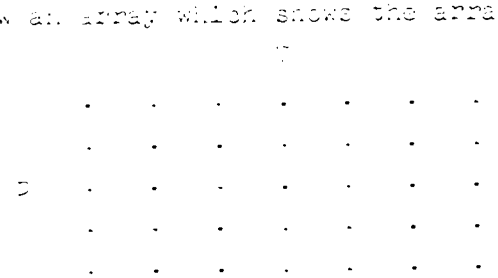


Let us begin with a set of 5 elements and a set of 4 elements and form an array to show their matchings.

How many elements are in the array? (20) With what two numbers did we begin? (5 and 4) What number is the result? (20) What operation are we using? (Multiplication)

Let us examine a problem in which we use an array and the operation of multiplication: "In a classroom, there are 5 rows of chairs. Each row has 7 chairs in it. How many chairs are in the room?"

Draw an array which shows the arrangement of the chairs.

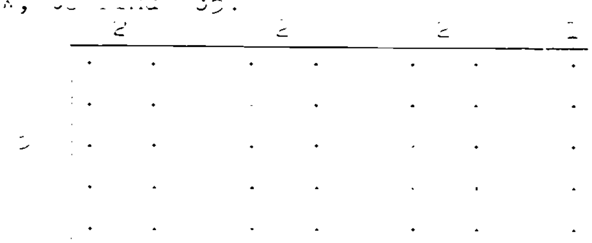


$5 \times 7 = n$

Write the mathematical sentence which expresses the operation on the two numbers, 5 and 7, and the result, n. (5 × 7 = n)

Do you know the number which n represents? (35) How did you find it was 35? (See following teacher note.)

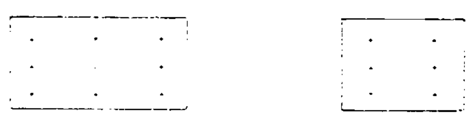
> The children should be encouraged to describe all of the ways they know of finding 5 × 7 including "knowing 5 × 7 = 35."
>
> They may separate the array as shown below, to find 35.



We can find the number of dots in the array, 5 × 7 = n in many ways. Some ways are easier than others: but once we know 5 × 7 = 35, we can put aside all other methods. Any time we have an array with 5 rows and 7 columns we know it has 35 elements. It is easier and quicker just to remember.

5 × 7 = 35 is a multiplication fact. You know many multiplication facts. What are some? (2 × 3 = 6; 5 × 6 = 30; etc.) Very soon we shall learn more of these facts which will make our work easier and faster.

> To find products which they do not know, the children should separate arrays by columns and use known products. For example: to find 3 × 5, we may separate a 3 by 5 array as follows:



3 × 3 = 9   3 × 2 = 6

> Thus 3 × 5 = (3 × 3) + (3 × 2) = 9 + 6 = 15. If 3 × 4 = 12 is known, the separation might be made at the last column, showing
>
> 3 × 5 = (3 × 4) + (3 × 1) = 12 + 3 = 15.

Caution: This way of writing is for the teacher and not to be given to the children at this time.

If we begin with a set of [illegible] elements and a set of [illegible]2 elements could we make an array showing their matchings? We would need a very large piece of paper and it would take a long, long time to make an array from the two sets.

We will learn to multiply numbers without making arrays. This will not require as much paper or as much time. This is the way mathematics helps us. It helps us to think about numbers in simple ways.

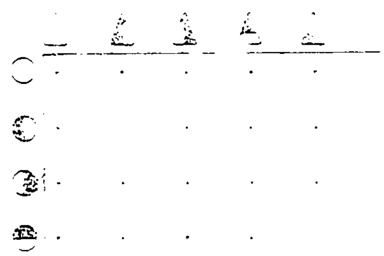
2[illegible]

MULTIPLICATION

An operation on numbers is a way of thinking about two numbers and getting one and only one number. When we think of 4 and 5 and get 20, we call this multiplication.

We use arrays to help us understand multiplication.

Where is an array which shows all the matchings of a set of 4 elements with a set of 5 elements? There are 20 matchings.



Some examples of these matchings are:

Notice 2 objects are needed to make 1 matching. Try to draw some other possible matchings.

The array gives us a picture of the mathematical sentence. $4 \times 5 = 20$. We read this "4 times 5 equals 20." We write the number of rows first and the number of columns second.

rows columns elements

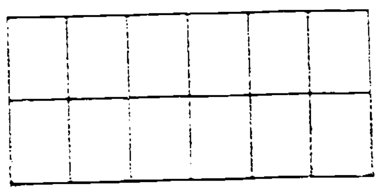
[illegible]

WRITING MATHEMATICAL SENTENCES

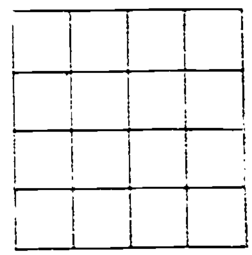
Exercise Set 2

Write the mathematical sentence which belongs with each of these arrays.

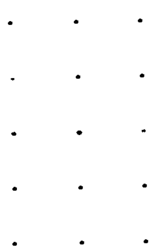
> In each activity, we have stated the number of rows first. Children should think in this way. For example, in exercise 1, there are 2 rows and 6 columns. The mathematical sentence must be 2 x 6 = 12.

1. 

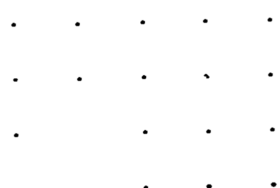
(2 x 6 = 12)

2. 

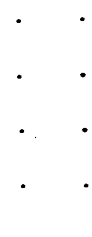
(4 x 4 = 16)

3. 

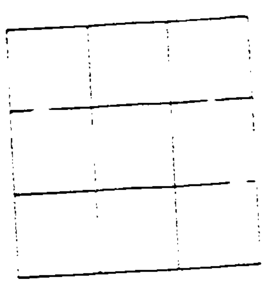
(5 x 3 = 15)

4. 

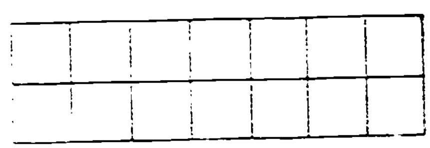
(4 x 5 = 20)

5. 

(5 x 2 = 10)

6. 

(3 x 3 = 9)

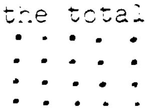
7. 

(2 x 7 = 14)

Exercise Set 4

> You may wish to make this exercise a class activity depending upon the level of the class.

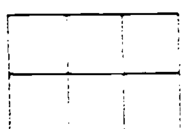
1. Each of 4 elementary schools has a basketball team of 5 players. Draw an array with one dot for each player. Using an array describe the number of players. Write the mathematical sentence. What does this sentence say about the total number of players on the team?



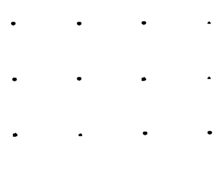
$4 \times 5 = 20$

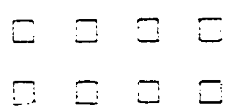
There are 20 players on the team.

Make a story problem which each of the following arrays helps to answer.

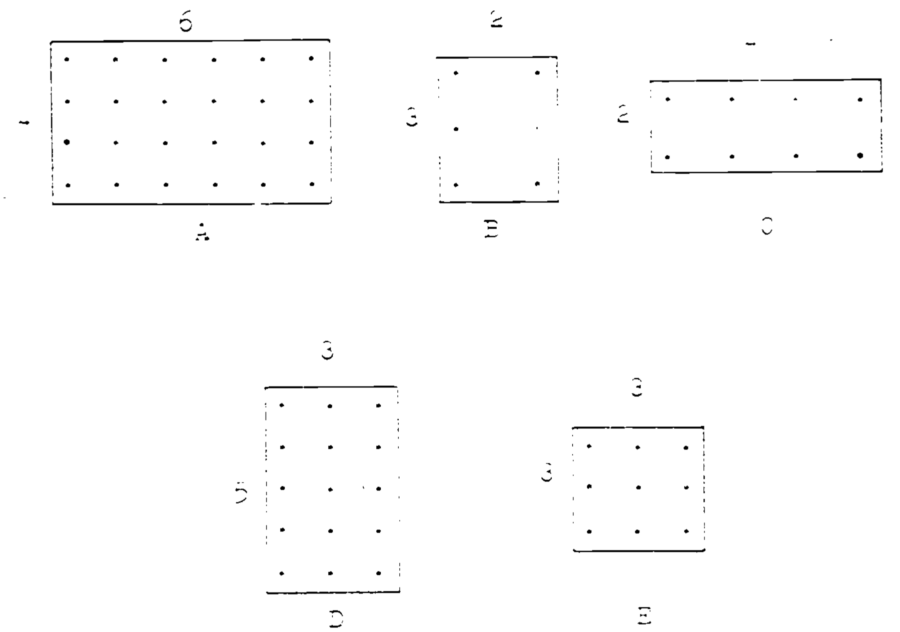
2. 

3.

4. 

5. 

You may want to analyze the following arrays before solving problems 6 through 10. Example: Array A has 4 rows, 6 columns, and 24 elements.

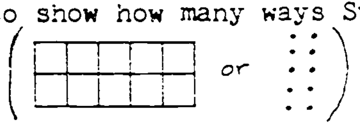


6. Three teams of Cole School are to play. Three teams from Grant School are to play. Which array above shows the number of games that can be arranged if each Cole team plays each Grant team just once? (E)

7. Mr. Smith is buying a car. The company offers tops in 5 colors and bodies in 3 colors. Which array above shows the number of possible color combinations? (D)

8. Mary has 4 dolls and 6 dresses. Which array above shows how many matchings of dolls and dresses Mary can make? (A)

9. Three children play violin and two other children play piano. Which array above shows how many duets can be played so that each violinist plays with each pianist? (B)

10. Sue has 2 watches and 5 different colored bands. Make an array to show how many ways Sue can match her watches and bands. ()

  a. How many rows must the array have? (2 or 5)

  b. How many columns must the array have? (5 or 2)

  c. How many ways can Sue match her watches and bands? (10)

11. Paul has 2 neckties and 3 colored handkerchiefs. Draw an array to show the different ways he can match the neckties and handkerchiefs. (• • • / • • •)

  a. How many rows has the array? (2)

  b. How many columns has the array? (3)

  c. How many matchings could Paul make? (6)

12. Linda has some necklaces and some bracelets. Here is an array which shows all the ways she can match her necklaces and her bracelets.

• • •

• • •

• • •

• • •

  a. If Linda has 4 necklaces, how many bracelets has she? (3)

  b. Write a mathematical sentence which belongs with the array. ($4 \times 3 = 12$)

  c. How many ways can Linda match her necklaces and bracelets? (12)

PROBLEMS

Exercise Set 5

Write a mathematical sentence which goes with each problem. Draw an array if you need one. Beginning with problem 4, be sure you **answer** each question in a complete sentence.

1. Write the mathematical sentence which shows the matchings of a set of 2 things with a set of 9 things. (2 X 9 = 18)

2. Write the mathematical sentence which shows the matchings of a set of 2 objects and a set of 3 objects. (2 x 3 = 6)

3. In how many arrays can 12 dots be arranged? (6)
   Write the mathematical sentences. (1 x 12 = 12, 12 x 1 = 12, 2 x 6 = 12, 6 x 2 = 12, 3 x 4 = 12, 4 x 3 = 12)

4. The calendar is arranged in 5 rows of squares. Each row is divided into 7 squares. How many squares are shown on the calendar?
   (There are 35 squares on the calendar.)

5. Some Christmas ornaments were packed in boxes of 4 rows. There were 3 ornaments in each row. How many ornaments were there in the box?
   (4 x 3 = n, n = 12, There were 12 ornaments in the box.)

6. A bar of chocolate candy was divided into 2 rows of 4 squares each. How many squares of chocolate were in the bar?
   (2 x 4 = n, n = 8, There were 8 squares of chocolate in the bar.)

7. There are 3 rows of windows in our room. Each row has 5 windows. How many windows are there in our room?

($3 \times 5 = w$
$15 = w$
There are 15 windows in our room.)

8. Candy was arranged in a box in 5 rows with 9 pieces of candy in each row. How many pieces of candy were in the box?

($5 \times 9 = c$
$45 = c$
There were 45 pieces of candy in the box.)

9. For our class picture, the children were grouped in 4 rows. There were 8 children in each row. How many children were there in the picture?

($4 \times 8 = p$
$32 = p$
There were 32 children in the picture.)

10. In a box, there were [illegible] rows of erasers [illegible] in each row. How many erasers were there in the box?

([illegible]
There were [illegible] erasers in the box.)

BRAINTWISTER: How many possible arrays of 24 dots could you make? (Draw them if necessary.) Describe each array by writing a mathematical sentence.

(There are 8 possibilities.
$1 \times 24 = 24$ $6 \times 4 = 24$
$2 \times 12 = 24$ $8 \times 3 = 24$
$3 \times 8 = 24$ $12 \times 2 = 24$
[illegible] $24 \times 1 = 24$)

HOW TO SHOW MULTIPLICATION

Objective: to help children learn and use the spoken and written symbols of multiplication

Vocabulary: Factor, product

Be sure to use the Exploration section as a class activity to develop the spoken and written symbols of multiplication.

HOW TO SHOW MULTIPLICATION

Exploration

When we think, talk, and write about the operation of addition, we have certain ways of indicating addition with words and other symbols.

Write a mathematical sentence which shows the addition of 4 and 5. $(4 + 5 = 9)$

How do you read this mathematical sentence?
(Four plus five is equal to nine.)

What are the numbers 4 and 5 called? (Addends)

What is the number 9 called? (Sum)

The sum, 9, is the result of operating on the addends, 4 and 5.

There is also a mathematical sentence to indicate multiplication. If the two numbers operated on are 4 and 5 and the result is 20, we can write the mathematical sentence

$$4 \times 5 = 20.$$

The numbers 4 and 5 are called factors of 20. The number 20 is called the product of 4 and 5.

The product, 20, is the result of operating on the factors 4 and 5.

Compare these two mathematical sentences.

a) $4 + 5 = s$ b) $4 \times 5 = p$

How are the sentences alike? (We begin with the same numbers, 4 and 5.)

How are the sentences different? (The results are different. In one sentence we use the operation of addition, and in the other we use the operation of multiplication.)

It is confusing to call the parts of the sent es by the same names because the operations are different. The 4 and 5 of sentence (b) must have special names. What are they called? (Factors of 20) What is 20 called? (Product of 4 and 5)

Here are some other mathematical sentences.

c) $5 \times 8 = 40$ d) $36 \times 424 = p$

What do we call the 5 in sentence (c)? (Factor)

What is the 8 called? (Factor)

What is the 40 called? (Product)

When we operate on two factors and get a product, we multiply.

What are the two factors in sentence (d)? (36 and 424)

What is the product? (p)

What number is the name of the product p ? (We don't know.)

We do not know now, but soon we will learn how to find it.

Summary

We write a multiplication sentence like this:

$$5 \times 4 = 20.$$

We read a multiplication sentence like this:

5 times 4 is equal to 20.

5 times 4 equals 20.

The names of the parts of a multiplication sentence are:

5	×	4	=	20
factor	times	factor	equals	product .

When we operate on <u>two</u> <u>factors</u> and get a <u>product</u>, we <u>multiply</u>.

USING ARRAYS IN MULTIPLICATION

Exercise Set 6

Each pair of numbers listed below shows the number of rows and the number of columns in an array. Write the mathematical sentence which tells how many elements are in each array.

Example: 5 , 3 $5 \times 3 = 15$

1.	4 , 2	$4 \times 2 = 8$	9.	2 , 5	$(2 \times 5 = 10)$
2.	4 , 3	$(4 \times 3 = 12)$	10.	3 , 5	$(3 \times 5 = 15)$
3.	4 , 4	$(4 \times 4 = 16)$	11.	4 , 5	$(4 \times 5 = 20)$
4.	5 , 4	$(5 \times 4 = 20)$	12.	5 , 5	$(5 \times 5 = 25)$
5.	6 , 4	$(6 \times 4 = 24)$	13.	5 , 6	$(5 \times 6 = 30)$
6.	7 , 4	$7 \times 4 = 28$	14.	5 , 7	$(5 \times 7 = 35)$
7.	8 , 4	$8 \times 4 = 32$	15.	5 , 8	$(5 \times 8 = 40)$
8.	9 , 4	$(9 \times 4 = 36)$	16.	5 , 9	$(5 \times 9 = 45)$

Make a chart with two columns as shown below. Complete your chart. An example is given.

Number of Rows and Number of Columns in Each Array	Mathematical Sentence Which Describes the Number of Elements in Each Array
Example 5 , 2	5 × 2 = 10
17. 6 , 3	6 × 3 = 18
18. 3 , 4	3 × 4 = 12
19. 7 , 5	7 × 5 = 35
20. 6 , 4	6 × 4 = 24
21. 8 , 5	40 = 8 × 5
22. 7 , 4	7 × 4 = 28
23. 6 , 5	6 × 5 = 30
24. 8 , 3	24 = 8 × 3
25. 5 , p	5 × p = 15
26. n , 3	n × 3 = 15
27. n , p	n × p = 15

MULTIPLICATION FACTS

Objective: To help children discover and learn the basic multiplication facts

Materials: Duplicate blank multiplication chart similar to the one on page 267

Read note between bars.

Exploration:

The basic multiplication facts include the 100 facts which result from finding the product for each pair of numbers from the set of numbers, 0 through 9.

The children should discover the facts in many ways and then learn them. The teacher should encourage the children to remember the facts, because knowing the facts is the easiest and quickest way of multiplying. Individualized instruction is needed to help each child learn these facts.

To discover new multiplication facts, the children will be asked to (1) separate an array into two arrays associated with known facts, then (2) add the number of elements in the two smaller arrays. Experiences with arrays should result in a more mature way of thinking of a multiplication fact. They may help many children to learn the facts. For example a child may think of 7 × 6 as (7 × 3) + (7 × 3) or (21 + 21) = 42.

Three types of arrays other than the many drawn on the chalkboard are needed for reference.

First, a teacher will find it helpful to have an array of 9 columns each of 2's, 3's, ... 9's for easy reference. It is best that these be large enough for demonstration purposes yet the size children can handle. An equally large plain piece of cardboard is also needed to cover sections of the arrays.

Second, a teacher will need the same arrays painted on fabric or sturdy paper. These arrays should be made so that they may be folded into sections when pupils discover a new fact or find a new way to show a fact. For example, an array of 9 columns of 6's may be folded into two arrays, one of 5 columns of 6's and one of 4 columns of 6's. The children may now add 30 and 24 to get 6 × 9 = 54.

Exploration:

The following exploration leads to the multiplication chart.

This array is one which you have always found easy. You know many things about columns of 5's.

```
.   .   .   .   .   .   .   .   .
.   .   .   .   .   .   .   .   .
.   .   .   .   .   .   .   .   .
.   .   .   .   .   .   .   .   .
.   .   .   .   .   .   .   .   .
```

Cover part of this array with cardboard. How many dots are in the array you see?

> The teacher should cover several columns of the array. Then cover one additional column at a time until all columns have been covered. Uncover the array one additional column at a time until all columns have been uncovered. In each instance the children should tell the number they see in the array. This should move quickly but be purposeful. It should be made clear by the teacher that the number of elements in the rows and columns forming the array are factors, and the number of elements in the array is the product.
>
> The teacher also should cover rows of the array one at a time and have the children identify the number of elements in the part of the array they can see.

Describe the array. (It has 5 rows of 9 dots each or 45 total.)

> The development here should follow that used for the discovery of the multiplication facts of 5's and 4's. The activities should be swift moving and varied so that no child finds them boring. The activities should be conducted with purpose so that they will help the children know and use the basic facts without hesitation.

[illegible]

[illegible]

[illegible] addition chart and a subtraction chart.
[illegible] a multiplication chart.

[illegible]

[illegible] and fill the column with [illegible] written in [illegible] column. Do it now.

[illegible] pairs of numbers and write their products in [illegible] chart is completed.

The teacher should suggest [illegible] ways [illegible] be [illegible] knowing the products so they may be recalled quickly when needed. The game of Buzz [illegible] motivate the child to learn the multiplication facts more quickly. Buzz may be used as an activity for the whole class but it is [illegible] effective with small groups. It may be varied by "buzzing" different multiples.

In the game, each person in turn says a number. The first person says 1, the next person says 2, etc. The group decides in advance which number to "buzz." If the number is 5 or a multiple of 5, the child whose turn it is responds with "buzz." The counting will be 1, 2, 3, 4, buzz, 6, 7, 8, 9, buzz, 11, 12, 13, 14, buzz, etc.

Before the next pupil page be sure all students know the meaning of the word digit. A digit is any one of the ten Hindu-Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. In the decimal numeration system, we refer to the units' digit, the tens' digit, etc. A decimal numeral like [illegible] is said to have three digits. Its units' digit is [illegible]. Its tens' digit is [illegible]; its hundreds' digit is [illegible].

MULTIPLICATION FACTS

Exploration

It is important for you to learn how to multiply numbers quickly and correctly. This will help you work exercises where an array can be made to picture the problem. Already you have learned some products by counting and using addition. Now, you must learn how to multiply two small numbers and how to remember these facts well without having to look at arrays. In multiplying we think about two numbers and get a product. You already know many products. Think of some of them. What products do you know as you look at this array? 2 × 8

. . . .

. . . .

Did you use all or part of the array? For some products I used part of the array and for one I used all of the array.

When you look at the part formed by a set of 2 rows and 8 columns, how many dots do you see?

When you look at the part of the array formed by a set of 2 rows and 7 columns, how many dots do you see?

When you look at the part of the array formed by a set of 2 rows and 1 column, how many dots do you see? Complete the following mathematical sentences.

2 × 1 = ___	2 × 4 = ___
2 × 8 = ___	2 × 3 = ___
2 × 7 = ___	2 × 2 = ___
2 × 6 = ___	2 × 1 = ___
2 × 5 = ___	

What kinds of things come in groups of 3? Do they form an array? [illegible handwriting]

Here is an array with 3 rows and 9 columns.

. . . .

. . . .

. . . .

How many dots are in the array? [illegible]

Cover one column. How many dots are in the array you see? (24)

Cover one additional column. How many dots are in the array you see? [illegible]

Using your 3 by 9 array, can you write all of the mathematical sentences as you did with the 2 by 9 array? (Yes)

Name some things which come in groups of 3. Do they form an array? [illegible handwriting]

We realize that we could not have an array without both rows and columns. However, as we work with the 3 by 9 array, we find that each time we cover a column, we are subtracting three elements from the total number of elements. Compare the information in the chart.

	3 × 9 = 27	You have a total of 27 elements.
Cover 1 column	3 × 8 = 24	27 − 3 = 24
Cover 2 columns	3 × 7 = 21	24 − 3 = 21
Cover 3 columns	3 × 6 = 18	21 − 3 = 18
Cover 4 columns	3 × 5 = 15	18 − 3 = 15
Cover 5 columns	3 × 4 = 12	15 − 3 = 12
Cover 6 columns	3 × 3 = 9	12 − 3 = 9
Cover 7 columns	3 × 2 = 6	9 − 3 = 6
Cover 8 columns	3 × 1 = 3	6 − 3 = 3
Cover 9 columns	3 × 0 = 0	3 − 3 = 0

By covering the last column you may imagine a 3 by 0 array (or pattern) or you may think of subtracting 3 elements. In either case there would be no (or zero) elements remaining. We know $3 - 3 = 0$, so we may assume $3 \times 0 = 0$.

> By following the same reasoning, we may develop the idea that $2 \times 0 = 0$. As we develop the multiplication facts, we will note that any number times zero will give a product of zero: $n \times 0 = 0$.
> It follows, by using the commutative property, that if $n \times 0 = 0$, then $0 \times n = 0$. $n \times 0 = 0 \times n$

Here is an array. Describe it. (4 by 9 array)

```
.  .  .  .  .  .  .  .  .

.  .  .  .  .  .  .  .  .

.  .  .  .  .  .  .  .  .

.  .  .  .  .  .  .  .  .
```

Using a piece of paper, cover the columns of the array one at a time. How many elements are there in the part you see? Begin with the whole array. Uncover the array one column at a time. Tell how many elements there are in the section you see each time.

Name some things which come in 4's and tell if they form an array? Using your 4 by 9 array, write your multiplication facts with factors of 4.

Exploration: Completing the multiplication table through 9 x 9.

> Do not expect all children will be learning the multiplication facts. It is not advisable to stop after the development of the multiplication chart and wait until all pupils have mastered the facts. Teachers will need to use extra practice sheets, flash cards, and other devices to aid children in acquiring immediate recall.

Let us take another look at our multiplication chart. Why is part of it still blank? (We have not discovered the multiplication facts belonging to that part of the chart.) Could you? (Yes)

How many spaces are there left to fill? (10) On the missing part of the chart, what multiplication pairs belong in the 6 row? (6 x 6, 6 x 7, 6 x 8, 6 x 9)

Notice that there is a factor 6 in each multiplication indicated in the first row. Make a 6 by 5 array. How many elements are in the array? (30) What would you need to do to the 6 by 5 array to make a 6 by 6 array? (Add one more column of dots.) How many elements are in the 6 by 6 array? (36) What would you need to do to the 6 by 6 array to make a 6 by 7 array? (Add one more column of dots.) How many elements are in the array? (42) Now add another column of 6 dots to the array. What size array have we now? (We have a 6 by 8 array with 48 elements.) Add another column of 6 dots to the array. What size array have we now? (We have a 6 by 9 array with 54 elements.) Your array should look like this:

```
.  .  .  .  .  .  .  .  .
.  .  .  .  .  .  .  .  .
.  .  .  .  .  .  .  .  .
.  .  .  .  .  .  .  .  .
.  .  .  .  .  .  .  .  .
.  .  .  .  .  .  .  .  .
```

Write all the mathematical [illegible] using the [illegible] array. [illegible] may complete [illegible].

Before completing the multiplication chart it is helpful to think about how a product may be found either by folding arrays. Thus, you may lead pupils to a greater understanding of multiplication relationships. For example, to find [illegible],

(1) Fold the array into 2 parts, one of [illegible] columns of [illegible], and one of [illegible] columns of [illegible]'s. Then the array will [illegible] and [illegible].

(2) Fold the array into 2 parts, the [illegible] columns of [illegible]'s, and one of [illegible]. Then the array will contain [illegible] and [illegible].

(3) Explore other possibilities.

What is the best method of showing that there are [illegible] dots in the array? (The best way is to use the multiplication fact [illegible].)

Using the same procedures suggested for the [illegible] array, [illegible] other facts. Complete the multiplication chart.

These last several facts are of increasing importance. The children find them more difficult to remember. They need to find the products in many ways, to say them, and to use them many times.

The teacher should help the children realize that it is easier and faster to know products by remembering rather than having to think of an array each time.

USING YOUR MULTIPLICATION CHART

Working Together

You have just completed the multiplication chart blank given you by your teacher. Using this chart, answer the following questions.

1. Why are all the products 0 in the first row and the first column? (Any number multiplied by zero is zero.)

2. Why are all the products in the second column the same as the numerals in the first column? Why are the products in the second row the same as the numerals across the top of the chart? (Any number multiplied by 1 is the number itself.)

3. Read the products from the 9 <u>column</u>.
   Read the products from the 9 <u>row</u>.

   a. Read in order the digits in the ones' place of these products. (9, 8, 7, 6, 5)

   b. How do the digits in the ones' place change? (The digits in the ones' place decrease by 1 beginning with 9 from the product 9 × 1.)

   c. Read in order the digits in the tens' place of the products you read. (1, 2, 3, 4)

   d. How do the digits in the tens' place change? (The digits in the tens' place increase by 1 beginning with 1 from the product of 9 × 2.)

4. What sum do you get if you add the numbers represented by the digits of each of the products in the 9 <u>column</u>? (9)

5. Complete the following statements by using either "always" or "not always."

   a. The product of two even numbers is always an even number.

   b. The product of two odd numbers is always an odd number.

   c. The product of an odd and an even number is always an even number.

6. The product of any number and one (n x 1) is n.

   The product of any number and zero (n x 0) is 0.

USING OPERATIONS

Exercise Set 1

Make a chart with 4 columns as shown below. Study and complete the chart.

Numbers Operated On	Number which Results	Operation Used	Mathematical Sentence Showing Operation Used
Examples:			
4 , 2	6	Addition	4 + 2 = 6
7 , 4	3	Subtraction	7 - 4 = 3
4 , 3	12	Multiplication	4 × 3 = 12
1. 12 , 5	7	Subtraction	12 - 5 = 7
2. 6 , 4	24	Multiplication	6 × 4 = 24
3. 7 , 5	35	Multiplication	7 × 5 = 35
4. 4 , 8	32	Multiplication	4 × 8 = 32
5. 7 , ___	7	Multiplication	7 × 1 = 7
6. 9 , 7	16	Addition	9 + 7 = 16
7. 8 , 5	40	Multiplication	8 × 5 = 40
8. 5 , 0	0	Multiplication	5 × 0 = 0
9. 5 , 3	15	Multiplication	5 × 3 = 15

Since fourth grade children may spend much time making a chart of this type, you may want to duplicate it for them.

Extending [illegible]

Sometimes it is difficult for children to see that multiplication is the operation to use when solving word problems. If children do not recognize that multiplication is indicated, it might be helpful to construct an array. For instance, consider problem [illegible] in Exercise Set 5. The problem stated is:

> There are 8 children planning to have a "cook out." Each child must bring 3 hot dogs. How many hot dogs will they have to cook, if they cook all the hot dogs?

You might ask pupils to draw an array to show that 8 children each brought 3 hot dogs. There are 2 possibilities.

```
. . . . . . . .          . . .
. . . . . . . .          . . .
. . . . . . . .          . . .
                         . . .
   3 x 8 = 24            . . .
                         . . .
                         . . .
                         . . .

                       8 x 3 = 24
```

In the first array, 3 hot dogs are represented by each column, but in the second array 3 hot dogs are represented by each row. Each row in the first array OR each column in the second array represents 8 children. We can see from the arrays that the boys will have to cook 24 hot dogs. 3 x 8 = 24 or 8 x 3 = 24.

Problem [illegible] deals with something more abstract. We will have to imagine that an orderly arrangement of dots on paper can represent minutes or round trips. The problem is:

> A boy took 9 minutes to ride his bicycle to the store and home again (round trip). If he made 6 round trips in one day, how many minutes did he spend between his home and the store?

The arrays should look like one of these:

```
                               . . . . . .
. . . . . . . . .              . . . . . .
. . . . . . . . .              . . . . . .
. . . . . . . . .              . . . . . .
. . . . . . . . .              . . . . . .
. . . . . . . . .              . . . . . .
. . . . . . . . .              . . . . . .
                               . . . . . .
   6 x 9 = 54                  . . . . . .

                               9 x 6 = 54
```

By counting the elements (or knowing the multiplication facts) in the 2 arrays, we see there are [illegible] elements in [illegible] array. This means that [illegible] × [illegible] = [illegible] and [illegible] × [illegible] = [illegible]. The boy spent [illegible] minutes between his home and the store.

PRACTICE IN MULTIPLICATION

Exercise Set 8

Complete the exercises 1 through 14 so they will form true mathematical sentences.

1. 5 × 7 = 35	8. 6 × 8 = 48
2. 6 × 2 = 12	9. 8 × 9 = 72
3. 4 × 6 = 24	10. 7 × 9 = 63
4. 6 × 7 = 42	11. 6 × 6 = 36
5. 6 × 9 = 54	12. 9 × 6 = 54
6. 7 × 7 = 49	13. 8 × 8 = 64
7. 7 × 8 = 56	14. 9 × 9 = 81

Write a mathematical sentence which goes with each problem. Solve it. Be sure you answer the question in a complete sentence.

15. There are 8 children planning to have a "cook out." Each child must bring 3 hot dogs. How many hot dogs will they have to cook, if they cook all the hot dogs?
    (8 × 3 = n or 3 × 8 = n
    24 = n They will have to cook 24 hot dogs.)

16. One candy bar costs 7¢. How much would 8 candy bars cost?
    (8 × 7 = n or 7 × 8 = n
    56 = n
    Eight candy bars would cost 56¢.)

17. A boy took 9 minutes to ride his bicycle to the store and home again (round trip). If he made 6 round trips in one day, how many minutes did he spend between his home and the store?
    (6 × 9 = n or 9 × 6 = n
    54 = n
    The boy spent 54 minutes between his home and the store.)

18. Jim swam one lap of the pool in 9 seconds. If he swam at the same rate, how long would it take him to swim 8 laps? (8 x 9 = l or 9 x 8 = l
72 = l
It would take Jim 72 seconds to swim 8 laps.)

19. There are 6 pairs of gym socks in a box. If a salesman has 8 boxes of gym socks, how many pairs will he have?
(8 x 6 = s or 6 x 8 = s 48 = s
A salesman will have 48 pairs.)

20. There are 9 rows of chalk in a box, and there are 7 sticks of chalk in each row. How many sticks of chalk are in the box? (9 x 7 = c
63 = c
There are 63 sticks of chalk in the box.)

THE COMMUTATIVE PROPERTY OF MULTIPLICATION

Objective: To help children understand and use the property of multiplication exemplified by [illegible] × [illegible] = [illegible] × [illegible] and stated by the formula: $a \times b = b \times a$

Exploration:

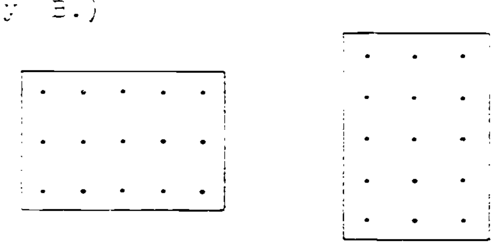
It is important for children to understand that in multiplication of whole numbers the order of factors may be changed without changing the product.

The fact that if a and b are whole numbers $a \times b = b \times a$, should be easy for children to understand. It is evident for the multiplication facts which children know. The importance of the commutative property is best shown later when the children are learning the algorism for multiplying numbers whose numerals have several digits.

The commutative property of addition will help the children to understand the commutative property of multiplication.

The arrays which the teacher and children have made and collected should be used in this exploration. They may be turned through a right angle so their rows and columns are interchanged. The children may describe this as turning an array on its side. This results in two multiplication facts for each array, e.g., $7 \times 8 = 56$ and $8 \times 7 = 56$.

Here are two arrays. Can you think of them as the same array? (Yes, I can turn Array A on its side. Then it is the same as Array B.)



Write the mathematical sentences suggested by the arrays. ($3 \times 5 = 15$, and $5 \times 3 = 15$)

When you thought about the two arrays as the same array, did you have to know there were 15 dots in each array before you could imagine they were the same? (No, their shape is the same. One has 5 rows and 3 columns: the other has 3 rows and 5 columns.) Can I make one mathematical sentence which shows the relationship of array A and array B? (Yes) What is it?

$$3 \times 5 = 5 \times 3$$

Now I am going to ask you to use your imaginations. Suppose C is a very large array. Pretend it has 67 rows and 59 columns. Suppose array D has exactly the same shape as C, only it is turned on its side. How many rows and columns must D have? Remember, it is exactly the same shape as C. (59 rows and 67 columns.) Could you write a mathematical sentence which shows that C and D have the same number of elements.

$$67 \times 59 = 59 \times 67$$

Do you have to find the unknown products 67 × 59 and 59 × 67 to be sure they are the same? (No, the shape is enough.) You may want to test these ideas with other arrays and products.

> The teacher should continue to have the children describe pairs of equal arrays in two positions (interchanging rows and columns), and write mathematical sentences suggested by the arrays. This should continue until there are several mathematical statements like the following on the chalkboard.
>
> $3 \times 5 = 5 \times 3$ $7 \times 9 = 9 \times 7$
>
> $3 \times [illegible] = [illegible] \times 3$ $10 \times 4 = 4 \times 10$
>
> $6 \times 4 = 4 \times 6$ $5 \times 10 = 10 \times 5$

Notice the statement you have made. How could you write a similar statement if an array were made from a set of 6 elements and a set of a elements? ($6 \times a = a \times 6$)

How could you write a similar statement if an array were made from a set of b elements and a set of c elements?

$$(b \times c = c \times b)$$

How could you write a similar statement if an array were made from a set of a elements and a set of b elements?

$$(a \times b = b \times a)$$

When we write 3×5 and 5×3, what operation is used? (Multiplication)

What are 3 and 5 called? (Factors) What name is used for the results? (Product)

Look again at the list of mathematical sentences we have made. What do you notice about the factors 3 and 5 in the mathematical sentence $3 \times 5 = 5 \times 3$? (They changed order.) Did the order of the factors change in $6 \times 5 = 5 \times 6$? (Yes.)

> The children should answer the same question for several of these sentences.

What happens to the product when the order of the factors is changed? Does it change? (No) Show this with an array and with a sentence. (From this array, I write $6 \times 7 = 42$. I turn it to form a different array and write $7 \times 6 = 42$. Since it is the same array, $6 \times 7 = 7 \times 6$.)

These sentences illustrate the <u>commutative property of multiplication</u>. Look at the multiplication chart you made. Find examples of the commutative property. The sentence $6 \times 7 = 7 \times 6$ is an application of $a \times b = b \times a$. We can find other applications of it on the chart.

Put a finger of one hand on the product of 6×7 and a finger of the other hand on the product of 7×6. Find other examples of the commutative property, e.g., $5 \times 4 = 4 \times 5$, etc., and place a finger of one hand on one product and a finger of the other hand on the other product.

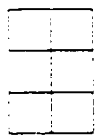
> The teacher should have the children find many examples like these, until they notice that the two products are in similar positions on opposite sides of a diagonal drawn from the top left corner to the lower right corner.

As you have pointed to these products, have you omitted any? (Yes, we have not touched 0, 1, 4, 9, ..., 81.) Why have you not touched these? (Their factors are the same. [illegible] 6 × 6. 49 = 7 × 7. We could not tell if the order of factors were changed.)

Is there any way you could fold the paper on which you made the table so that products having the same factors would fall on each other? Try it. Does 42 fall on 42, 20 fall on 20, etc. (Yes, if I do that I fold it along the line where 0, 1, 4, ..., 81 are.) Do the equal products fall on each other? (Yes, the chart is like a mirror. One section reflects the other.)

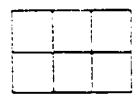
THE COMMUTATIVE PROPERTY OF MULTIPLICATION

A 3 by 2 array can be turned to form a 2 by 3 array.



3 by 2 Array

$3 \times 2 = 6$



2 by 3 Array

$2 \times 3 = 6$

This shows $2 \times 3 = 3 \times 2$.

A 78 by 65 array can be turned to form a 65 by 78 array. This shows

$$65 \times 78 = 78 \times 65.$$

When we write 65×78 in place of 78×65, we are using the <u>commutative property of multiplication</u>. We can use the commutative property to reduce the number of multiplication facts we must remember.

Exercise Set 2

1. a. Look at the product in each of the following mathematical sentences.

   $3 \times 5 = m$ $\qquad$ $5 \times 3 = n$

   b. Does $m = n$? Yes

   c. Using the two mathematical sentences in (a), make one sentence. ($3 \times 5 = 5 \times 3$)

2. Decide if each of the following statements is true or false. Write T if a statement is true. Write F if a statement is false.

   a. $(4 + 3) = (3 + 4)$ (T) $\quad$ d. $(6 \div 9) = (9 \div 6)$ (F)

   b. $(7 \times 4) = (4 \times 7)$ (T) $\quad$ e. $(5 \times 8) = (8 \times 5)$ (T)

   c. $(12 - 5) = (5 - 12)$ (F) $\quad$ f. $(4 - 10) = (10 - 4)$ (F)

3. Look at your answers to problem (2) and answer these questions:

   a. Does addition have the commutative property? (Yes)

   b. Does multiplication have the commutative property? (Yes)

   c. Does subtraction have the commutative property? (No)

USING THE NUMBER LINE

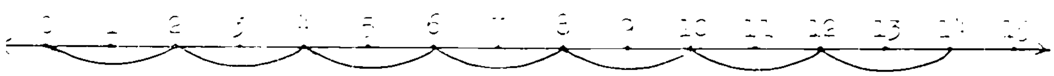
Objective: To provide the children with another approach to understanding the idea of multiplication

Materials: A number line of durable material with the whole numbers from 0 through 100

Exploration:

> The teacher should have a number line prepared on cardboard. Place the number line immediately at the top of the chalkboard so that lines similar to the ones below may be drawn by either pupil or teacher to make particular points in the discussion.
>
> We will find as we count by 2's that a definite rhythm will develop. The children should notice that each number considered will have the factor 2. These numbers which have a factor 2 are called multiples of 2.

Here is a number line with the whole numbers from 0 through 100. Notice that if we count by 2's, starting with zero, we will mark only those numbers which are multiples of 2. (0, 2, 4, 6, 8, 10, 12, 14, ...)



Now let us try counting by 3's using the number line. Again, notice that the numerals marked represent numbers which are multiples of 3. (0, 3, 6, 9, 12, 15, ...)

> After the children have reviewed the number line using those facts they know, develop counting by 6's, 7's, 8's, and 9's, using the number line and beginning with zero. The teacher should use the following procedure.

On the number line show us products which have as factor, 6. Beginning with zero, say the product as you point to the number line. (0, 6, 12, 18, 24, ... 54, ... 96) If the number line were extended, what product would follow 96? (102)

Have the children repeat this activity so that the rhythm of intervals of 6 may be noticed, and so that the children will become familiar with the products which have a factor 6. These are the multiples of 6.

Using the number line, finish these mathematical sentences. Try to remember the products.

6 × 9 = 54	7 × 9 = 63	8 × 9 = 72	9 × 9 = 81
6 × 8 = 48	7 × 8 = 56	8 × 8 = 64	9 × 8 = 72
6 × 7 = 42	7 × 7 = 49	8 × 7 = 56	9 × 7 = 63
6 × 6 = 36	7 × 6 = 42	8 × 6 = 48	9 × 6 = 54
6 × 5 = 30	7 × 5 = 35	8 × 5 = 40	9 × 5 = 45
6 × 4 = 24	7 × 4 = 28	8 × 4 = 32	9 × 4 = 36
6 × 3 = 18	7 × 3 = 21	8 × 3 = 24	9 × 3 = 27
6 × 2 = 12	7 × 2 = 14	8 × 2 = 16	9 × 2 = 18
6 × 1 = 6	7 × 1 = 7	8 × 1 = 8	9 × 1 = 9
6 × 0 = 0	7 × 0 = 0	8 × 0 = 0	9 × 0 = 0

You may want to use the game of buzz again. For example, begin by buzzing seven. You may want to use several techniques. First, count 1, 2, 3, 4, 5, 6, buzz, 8, 9, 10, 11, 12, 13, buzz, etc. Then you may want to buzz also those numbers which have 7 as a digit such as 17, 27, etc. If the group is extremely alert, you may include those numbers whose digits sum to 7, such as 16, 25, 34, etc. Now the game would sound like this: 1, 2, 3, 4, 5, 6, buzz, 8, 9, 10, 11, 12, 13, buzz, 15, buzz, buzz, 18, 19, 20, buzz, 22, 23, 24, buzz, 26, buzz, buzz, 29, 30.

COMPARING PRODUCTS

Exercise Set 10

Make each of the following mathematical sentences true by completing it with one of the symbols >, <, or =.

1. 1×12 = 4×3
2. 4×4 < 4×5
3. 2×6 = 4×3
4. 6×4 = 3×8
5. 3×5 < 4×8
6. 3×8 = 6×4
7. 7×4 > 9×3
8. 6×8 = 8×6
9. 9×4 = 6×6
10. 9×5 < 6×8
11. 7×5 > 7×3
12. 6×3 = 9×2
13. 9×3 < 8×7
14. 5×3 < 6×7
15. 7×8 > 5×7
16. 7×9 > 8×1
17. 5×9 < 7×8
18. 6×7 > $20 + 20$
19. 8×9 > 8×8
20. $94 - 40$ = 6×9
21. 9×9 > 8×9
22. 8×8 > 7×7
23. 8×7 < 10×6
24. $8 \times n$ > $2 \times n$, where $n \neq 0$
25. $n \times 6$ = $n \times 8$, where $n = 0$
26. $8 \times n$ > $7 \times n$, where $n \neq 0$
27. $n \times 4$ > $n + 4$, where $n > 1$
28. $n \times 4$ < $n \times 5$, where $n > 0$

BRAINTWISTER: 29. 8×8 > 9×7 > 8×4

30. $8 + 5$ < 7×4 < 5×9

23

[illegible]

Objectives: To help children understand that the operation on numbers called division can be described as finding an unknown factor when the product and one factor are known.

To lead children to conclude that if they can multiply, they can divide.

Vocabulary: Division

> You will notice that a study of problems based upon the multiplication facts with no use of division signs precedes introduction of the usual notation.
>
> We can picture division with arrays. If you feel this would be of value to your class, the following is a suggested procedure.

The mathematical sentence, 3 x ☐ = 12, indicates that we have 12 objects arranged in 3 rows. We must find the number of columns. Let's construct an array on the chalkboard (or feltboard). We will take 12 objects and arrange the first column thus, ⁝ Now we'll add the second column ⁝ ⁝ . So far we have used six objects. By adding a column at a time we can see that a 3 by 4 array with 12 objects looks like this.

. . . .

. . . .

. . . .

There are 4 columns. This means ☐ = 4 in our mathematical sentence, 3 x 4 = 12.

If the sentence were ☐ x 4 = 12, we would have 4 columns and need to find the number of rows. Since we have 4 columns we start with the first row. . . . . Now add the second and third rows. The array would look like this.

. . . .

. . . .

. . . .

We can see that we would use 3 rows in showing 12 objects arranged in 4 columns. 3 x 4 = 12.

Exploration:

Earlier in this unit, we developed a multiplication chart showing all the multiplication facts concerning whole numbers through $9 \times 9 = 81$. Now let us refer to this chart and review its structure. We can find answers to questions involving factors and products by looking at it.

> With such examples as $3 \times [illegible] = [illegible]$, $5 \times [illegible] = [illegible]$, $[illegible] \times [illegible] = [illegible]0$, review the procedure for finding these answers on the chart.

Looking at the multiplication chart, can we find all products (other than zero) of equal factors? (Yes, $1 \times 1 = 1$, $2 \times 2 = 4$, ..., $9 \times 9 = 81$.) Can you make an array that shows these facts? (Yes) What do you notice about each array of this type? (Each array has the same number of rows as columns.) If you made the array using squares, what shape would your array form? (The array would form a square with the same number of squares in the rows as we have in the columns.) We can say that numbers like 1, 4, 9, 16, ... are "square numbers" or "squares". Why? (The factors are the same. Thus they have the same number of rows and columns making a square array.) We can say at this point that "4 is the square of 2", "25 is the square of 5", etc. What is the square of 3? (9) What is the square of 6? (36)

Referring to the multiplication chart, make another chart (as follows) showing the numbers through 9 and their squares.

Number	1	2	3	4	5	6	7	8	9
Square	1	4	9	16	25	36	49	64	81

FINDING UNKNOWN FACTORS

Exploration

Earlier we talked about an operation as thinking about two numbers and getting a third number. If we think about 4 and 6 and get 24, we are using multiplication. We can write the mathematical sentence:

$$4 \times 6 = 24.$$

What are the numbers to be operated on? (4 and 6) What is the result. (24) What operation was used? (Multiplication)

Look at these mathematical sentences.

$3 \times 5 = p$	and	$3 \times n = 15$
$7 \times 2 = q$		$m \times 2 = 14$

How are these two sets of mathematical sentences different? (The first two have unknown products. The other two have unknown factors.)
How are the four sentences alike? (All four sentences show multiplication)

Do you multiply the two factors to find the unknown number in the sentence $3 \times 5 = p$? (Yes) Do you multiply the two factors to find the unknown number in the sentence $7 \times 2 = q$? (Yes)

Do you multiply the two factors to find the unknown number in the sentence $3 \times n = 15$? (No) Do you know what number n represents? (Yes, $n = 5$) How do you know? (I know because $3 \times \underline{5} = 15$) Do you know what number m represents in the mathematical sentence $m \times 2 = 14$? (Yes, $m = 7$) How do you know? (I know $\underline{7} \times 2 = 14$)

In the sentence $5 \times r = 10$, to find r, ask yourself what factor times 5 is 10.

Let us try some others.

$$6 \times n = 12$$

$$t \times 3 = 15$$

What number does n represent? (2) How do you know? (I know $6 \times \underline{2} = 12$.)

What number does t represent? How do you know?
(I know $\underline{5} \times 3 = 15$.)

> It is important for children to realize that complete mastery of the multiplication facts is necessary. The better they know these facts, the easier division will be for them. We need a good understanding of the multiplication facts in order to multiply and divide. We can do both operations with one set of facts.

Exercise Set 11

Find the unknown factor in the following sentences.

Example: $6 \times n = 24$

$n = 4$

1. $n \times 8 = 24$ $n = 3$
2. $2 \times t = 16$ $t = 8$
3. $p \times 4 = 16$ $p = 4$
4. $n \times 2 = 16$ $n = 8$
5. $p \times 6 = 48$ $p = 8$
6. $5 \times p = 45$ $p = 9$
7. $n \times 3 = 18$ $n = 6$
8. $8 \times n = 0$ $n = 0$

BRAINTWISTERS:

9. $n \times n = 25$ $n = 5$
10. $n \times m = 23$ $n = 1,\ m = 23$ or $n = 23,\ m = 1$
11. $(3 \times n) \times 2 = 24$ $n = 4$
12. $765 \times n = 7,650$ $n = 10$
13. $(n \times n) \times 4 = 36$ $n = 3$
14. $(n \times n) \times n = 27$ $n = 3$
15. $p \times p = p$ $p = 0$ or 1
    (Here everything is missing!)

PROBLEMS

Exercise Set 12

Write a mathematical sentence which goes with each problem. Find the unknown factor. Draw an array if you need one. Be sure you answer the question in a complete sentence.

Example: Arrange 24 chairs in rows of 6 chairs each.
How many rows will there be?

```
.  .  .  .  .  .
.  .  .  .  .  .
.  .  .  .  .  .
.  .  .  .  .  .
```

$n \times 6 = 24$

$n = 4$

There will be 4 rows.

1. A class of 32 children was divided into groups of 8 for square dancing. How many dance squares were there?
   ($n \times 8 = 32$ or $8 \times n = 32$ There were 4 dance squares.)
   $n = 4$
2. Arrange 15 boys in 3 equal teams for a relay race. How many boys will there be on each team?
   ($b \times 3 = 15$ or $3 \times b = 15$ There will be 5 boys on each team.)
   $b = 5$
3. Mary is filling 3 Easter baskets. She has one dozen colored eggs. How many eggs can she put in each basket?
   ($3 \times n = 12$ or $n \times 3 = 12$ She can put 4 eggs in each basket.)
   $n = 4$
4. Bob arranged his collection of 28 butterflies in 7 rows. How many butterflies did he have in each row?
   ($7 \times n = 28$ Bob had 4 butterflies in each row.)
   $n = 4$
5. In a card game 32 cards were arranged face-up in rows with 4 cards in each row. How many rows were there?
   ($4 \times c = 32$ There were 8 rows.)
   $c = 8$
6. The 36 children who direct traffic were divided into squads of 6. How many squads were there?
   ($6 \times s = 36$ or $s \times 6 = 36$ There were 6 squads.)
   $s = 6$
7. How many weeks are there in 35 days?
   ($7 \times m = 35$ or $m \times 7 = 35$ There are 5 weeks in 35 days.)
   $m = 5$

LEARNING ABOUT DIVISION

Exploration

Look at these mathematical sentences.

(a) $5 \times 2 = n$ (b) $5 \times n = 10$

What is n in the sentence $5 \times 2 = n$? (10)

How do you know? (I know the multiplication fact $5 \times 2 = 10$.)

What is n in the sentence $5 \times n = 10$? (2)

How do you know? (I know that $5 \times 2 = 10$.)

What operation is used to find a product? (Multiplication)

In the sentence $5 \times n = 10$, n is an unknown factor. This operation of thinking about a number and one of its factors and finding an unknown factor is called <u>division</u>.

To divide we think what number times the known factor gives the product, or the known factor times what number gives the product.

Look at these mathematical sentences.

(a) $5 \times 8 = n$

(b) $5 \times r = 40$

(c) $p \times 3 = 21$

In the sentence $5 \times 8 = n$, 5 and 8 are factors, what is n called? Product

What operation is used to find n in the sentence $5 \times 8 = n$? (Multiplication)

What number is n? (40)

In the sentence $5 \times r = 40$, 5 is a known factor of the product 40. What is r called? (Unknown factor)

What operation is used to find r in the sentence $5 \times r = 40$? (*Division*)

What number is r? (*8*)

What did you think to find r in the sentence $5 \times r = 40$? (*$5 \times \underline{8} = 40$ $r = 8$*)

In the sentence $p \times 3 = 21$, 3 is a known factor of the product 21. What is p called? (*Unknown factor*)

What operation is used to find p in the sentence $p \times 3 = 21$? (*Division*)

What number is p? (*7*)

What did you think to find p in the sentence $p \times 3 = 21$? (*$\underline{7} \times 3 = 21$ $p = 7$*)

Division is not always written as $5 \times n = 15$, or $n \times 5 = 15$ or $15 = n \times 5$. These sentences show multiplication.

This same relationship may be stated as a mathematical sentence which shows division. It may be written $n = 15 \div 5$. This sentence is read "n equals 15 divided by 5". n is the <u>unknown</u> factor; 15 is the <u>product</u>, and 5 is the <u>known factor</u>.

How may we describe division? (*Division is used to find an unknown factor in a multiplication problem.*)
What must be given to find an unknown factor? (*We must know the product and one factor. We find the unknown factor by dividing.*)
Can you find the unknown factor in these?

$7 \times m = 14$ $\quad$ $p \times 5 = 35$ $\quad$ $40 = 8 \times q$
(*Yes, $m = 2$ $\quad$ $p = 7$ $\quad$ $q = 5$*)
How did you know these? (*I know $7 \times \underline{2} = 14$, etc.*)

You thought of a multiplication fact in order to divide. If you can multiply, you can divide. This saves time.

For each multiplication fact you know, you can find some division facts. Let's try it. State some division facts you know because

$6 \times 8 = 48$ $\quad 48 \div 6 = 8 \quad 48 \div 8 = 6$

$5 \times 9 = 45$ $\quad 45 \div 9 = 5 \quad 45 \div 5 = 9$

$8 \times 9 = 72$ $\quad 72 \div 8 = 9 \quad 72 \div 9 = 8$

Write each of these mathematical sentences as a mathematical sentence showing division.

a. $6 \times n = 24$ $\quad n = 24 \div 6$

b. $a \times n = 12$ $\quad n = 12 \div a$

c. $a \times n = b$ $\quad n = b \div a$

Summary

When we think of 6 and 2 and get 3, we are dividing. When we think of a number and one of its factors and get the other factor, we are dividing.

There are two ways to suggest division with mathematical sentences.

1. We can suggest division by a multiplication sentence with an unknown factor:

2	$\times$	n	=	6
(factor)		(unknown factor)		(product)

or

n	$\times$	2	=	6
(unknown factor)		(factor)		(product)

2. We also can suggest division by a mathematical sentence such as this:

$$6 \div 2 = n$$

We read this sentence

6 divided by 2 equals n .

If we think of 6 and 2 and get 3, we can write

$$6 \div 2 = 3.$$

Exercise Set 12

For each multiplication fact below, write two division facts:

Example: $8 \times 6 = 48$ $\quad$ $48 \div 6 = 8$
$48 \div 8 = 6$

1.	$9 \times 8 = 72$	$72 \div 8 = 9$	$72 \div 9 = 8$
2.	$7 \times 9 = 63$	$63 \div 9 = 7$	$63 \div 7 = 9$
3.	$6 \times 9 = 54$	$54 \div 6 = 9$	$54 \div 9 = 6$

Rewrite each multiplication sentence as a division sentence. Find the unknown factor.

Example: $7 \times n = 28$ $\quad$ $28 \div 7 = n$
$4 = n$

4.	$5 \times n = 25$	$25 \div 5 = n$, $n = 5$	10.	$p \times 6 = 48$	$48 \div 6 = p$, $p = 8$
5.	$n \times 8 = 24$	$24 \div 8 = n$, $n = 3$	11.	$8 \times q = 48$	$48 \div 8 = q$, $q = 6$
6.	$2 \times t = 16$	$16 \div 2 = t$, $t = 8$	12.	$n \times 3 = 18$	$18 \div 3 = n$, $n = 6$
7.	$p \times 4 = 16$	$16 \div 4 = p$, $p = 4$	13.	$8 \times n = 0$	$0 \div 8 = n$, $n = 0$
8.	$n \times 1 = 16$	$16 - 1 = n$, $n = 16$	14.	$n \times 5 = 25$	$25 \div 5 = n$, $n = 5$
9.	$9 \times n = 72$	$72 \div 9 = n$, $n = 8$	15.	$8 \times n = 64$	$64 \div 8 = n$, $n = 8$

Exercise Set 13

Mathematical Sentence	Operation Used	Unknown Number
Example: $5 \times 8 = n$	$\times$	40
1. $5 \times p = 40$	$\div$	8
2. $15 - 13 = q$	$-$	2
3. $16 \div$ [illegible] $= r$	$\div$	[illegible]
4. $5 \times n = 40$	$\div$	8
5. $43 = n + 20$	$-$	23
6. $p = 6 \times 9$	$\times$	54
7. $35 - n = 12$	$-$	23
8. $t \times 9 = 81$	$\div$	9
9. $48 \div 6 = n$	$\div$	8
10. $56 \div 8 = y$	$\div$	7
11. $75 + 28 = n$	$-$	103
12. $49 \div 7 = n$	$\div$	7
13. $49 - 7 = n$	$-$	42
14. $49 + 7 = n$	$-$	56

USING A MULTIPLICATION CHART TO DIVIDE

Working Together

In the multiplication chart, the numbers at the top and left side are factors. Numbers in the body of the table are products.

The multiplication chart can be very useful in finding division facts. To find n in $48 \div 6 = n$, for example, you may use either of the following ways.

(1) Think $6 \times n = 48$. Begin with row "6", and follow row "6" until you come to 48. Notice that 48 falls in column "8". Thus $6 \times 8 = 48$, so $48 \div 6 = 8$.

(2) Think $n \times 6 = 48$. Begin with column "6". Follow column "6" down until you come to 48. Notice that 48 falls in row "8". Thus $8 \times 6 = 48$, so $48 \div 6 = 8$.

Use your multiplication chart to find the unknown number in each of these division sentences.

(a) $42 \div 6 = n$ (c) $54 \div 9 = r$

(b) $72 \div 8 = p$ (d) $63 \div 7 = t$

> CAUTION: We have said that $m \div n$ names the only number p such that $p \times n = m$. Therefore $6 \div 0$ should name the only number p such that $p \times 0 = 6$. Since there is no such number, $6 \div 0$ is meaningless. Also, since there are many numbers n such that $n \times 0 = 0$, then we agree that the symbol $0 \div 0$ does not name any number.

Exercise Set 1-

Use the multiplication chart to find the unknown number.

1. Complete the following division sentences using the chart.

a. $72 \div 9 =$ 8	f. $18 \div 2 =$ 9
b. $63 \div 7 =$ 9	g. $32 \div 8 =$ 4
c. $45 \div 5 =$ 9	h. $28 \div 7 =$ 4
d. $56 \div 8 =$ 7	i. $64 \div 8 =$ 8
e. $81 \div 9 =$ 9	j. $48 \div 6 =$ 8

The numeral 4 appears on the multiplication chart 3 times. A different mathematical sentence goes with 4 each time it appears. These mathematical sentences are:

$$1 \times 4 = 4$$

$$2 \times 2 = 4$$

$$4 \times 1 = 4$$

2. How many times does 36 appear on the chart? (three times)

   Write the mathematical sentences which go with 36.

   Be sure you state the number of rows first. ($4 \times 9 = 36$, $9 \times 4 = 36$, $6 \times 6 = 36$)

3. How many times does 24 appear on the chart? (four times)

   Write the mathematical sentences which go with 24. ($3 \times 8 = 24$, $8 \times 3 = 24$, $6 \times 4 = 24$, $4 \times 6 = 24$)

4. How many times does 47 appear on the chart? Why?

   (It does not appear. It does not have two whole number factors other than itself and 1.)

RELATION OF MULTIPLICATION AND DIVISION

Objective: To help children understand that multiplication and division will "undo" each other

> You will remember that dividing by a number (not 0) is the inverse of multiplying by the same number. It is recommended that the term "inverse" not be used with fourth grade children. However, the idea is important for the meaning of the operations. The object of the exploration is to help children discover the relationship between multiplication and division.
>
> Introduce at this time the idea that an expression such as [illegible] × [illegible] = 12 may be read either as "[illegible] times [illegible] equals 12," or "[illegible] multiplied by [illegible] equals 12." Give some practice in reading similar expressions both ways.

Materials: [illegible] array for demonstration

Suggestions:

[illegible] the two problems which are on the chalkboard. For [illegible] the mathematical sentence.

(a) There were [illegible] rows of [illegible] chairs in a room. There were [illegible] people in [illegible] row. How many people were seated in [illegible]	(b) There were 48 people seated in rows with 6 people in each row. How many rows of people were there?
[illegible]	n × 6
[illegible] × [illegible]	so, n = 48 ÷ 6

[illegible] an array to show problem (a). Describe it. (There are [illegible] rows of [illegible] dots.) How many dots are in the array? (48) [illegible]

[illegible] the same array to show problem (b)? (Yes, we [illegible] rows with [illegible] dots in each row.)

To find [illegible] in problem (a) what operation did we use? ([illegible] multiplied [illegible] by [illegible]. The product was 48.) To find n in problem (b) what operation did we use? ([illegible] divided by 48 [illegible])

Did we [illegible] × 6 = [illegible] (Yes)

This same thing happens when you work with addition and subtraction. You find 2 + 7 = 9. Then you find that 9 − 7 = 2. You are back to 2. How did we describe this relationship of addition and subtraction? (We said that they undo one another (7 − 5) + 5 = 7 or (8 + 3) − 3 = 8.)

Do multiplication and division undo each other? (Yes) Let's try some exercises and see.

$$(8 \times 3) \div 3 = 8$$

$$(12 \div 3) \times 3 = 12$$

$$(9 \times 4) \div 4 = 9$$

The children should explain each of the above, e.g., 8 × 3 = 24, 24 ÷ 3 = 8. Also they should give other examples of multiplication and division undoing each other and review addition and subtraction undoing each other.

RELATION OF MULTIPLICATION AND DIVISION

Multiplication will undo division. Think of 8, divide by 2, and then multiply by 2. The result is 8. The <u>multiplication</u> by 2 <u>undid</u> the <u>division</u> by 2.

$$(\underline{8} \div 2) \times 2 = \underline{8}$$

Division will undo multiplication. Think of 8, multiply by 2, and then divide by 2. The result is 8. The division by 2 undid the multiplication by 2.

$$(\underline{8} \times 2) \div 2 = \underline{8}$$

Exercise Set 15

1. Copy and complete the table below.

DO	UNDO
Examples: $2 \times 3 = 6$	$6 \div 3 = 2$
$12 \div 3 = 4$	$4 \times 3 = 12$
a. $1 + 4 = 5$	$5 - 1 = 4$ or $5 - 4 = 1$
b. $5 - 2 = 3$	$3 + 2 = 5$
c. $n \times 6 = 24$	$24 \div 6 = n$ $24 \div n = 6$
d. $15 \div 5 = n$	$n \times 5 = 15$
e. $5 \times n = 10$	$10 \div n = 5$ $10 \div 5 = n$

2. For each mathematical sentence, tell what operation is used to find n.

   a. $2 \times n = 8$ (Division)
   b. $n \div 4 = 4$ (Multiplication)
   c. $n = 9 - 3$ (Subtraction)
   d. $28 = n \times 4$ (Division)
   e. $7 \times n = 42$ (Division)
   f. $36 \div 6 = n$ (Division)
   g. $48 - n = 8$ (Subtraction)
   h. $27 = n \times 9$ (Division)

Write a mathematical sentence which goes with each problem. Find the unknown factor. Be sure you answer the question in a complete sentence.

3. a. A checkerboard has 8 rows with 8 squares in each row. How many squares are there on a checkerboard?
   ($8 \times 8 = n$, $64 = n$ There are 64 squares on the checkerboard.)

   b. There are 64 squares on a checkerboard. There are 8 squares in a row. How many rows of squares are there?
   ($n \times 8 = 64$, $n = 8$ There are 8 rows of squares.)

4. a. An abacus shows 5 beads on each of 4 wires. How many beads are shown on the abacus?

(5 × 4 = b or 4 × 5 = b; b = 20 — There are 20 beads shown on the abacus)

b. 20 beads are divided equally on each of the 4 wires of an abacus. How many beads are on each wire?

(4 × w = 20 or w × 4 = 20; w = 5 — There are 5 beads on each wire.)

5. a. In the library, the card catalogue has 4 rows of drawers with 6 drawers in each row. How many drawers are in the card catalogue?

(4 × 6 = d; d = 24 — There are 24 drawers in the card catalogue.)

b. The 24 drawers of the card catalogue in the library are divided into 4 rows. How many drawers are there in each row?

(4 × r = 24; r = 6 — There are 6 drawers in each row.)

6. a. Bill had 30 heads of lettuce in his garden. They were evenly divided into 5 rows. How many heads of lettuce were there in each row?

(5 × h = 30; h = 6 — There are 6 heads of lettuce in each row.)

b. Bill had 5 rows of lettuce in his garden. There were 6 heads of lettuce in each row. How many heads of lettuce did Bill have in his garden?

(5 × 6 = l; l = 30 — Bill had 30 heads of lettuce in his garden.)

PROPERTIES OF ONE AND ZERO

OBJECTIVE: To help children understand the special properties of one and zero in multiplication and division.

Exploration:

Let us think about the mathematical sentence which is on the chalkboard and learn something about factors.

$$a \times n = 12$$

What whole numbers could a and n name if this sentence is to show a multiplication fact? Let us write them.

a	n	product
12	[illegible]	12
[illegible]	[illegible]	12
[illegible]	[illegible]	12
3	[illegible]	12
[illegible]	[illegible]	12
1	12	12

How large can the factor a be? (12) How small can it be? (1)

We can say that 12 is a factor of 12 because $12 = 12 \times 1$.

Also 6 is a factor of 12 because $12 = 6 \times 2$.

Do you know any other factors of 12? (Yes, 4, 3, 2, and 1 are also factors of 12.)

Is 5 a factor of 12? (No, there is no whole number n so that $12 = 5 \times n$.) Do you know other whole numbers which are not factors of 12? (Yes, 7, 8, 9, 10, 11, 13, 14, ...)

Give a factor of 15. (5) Give some others. (3, 15, and 1)

Give the factors of 24. (1, 2, 3, 4, 6, 8, 12, 24)

Look at our chart of factors of 12. Is 0 a factor of 12? (No, the reason is because $0 \times n = 0$. $0 \times n$ can never equal 12.)

Can 0 be a factor of 20? (No, the reason is because $0 \times n = 0$. $0 \times n$ can never equal 20.)

Can 0 be a factor of any whole number? (It can be a factor only of 0.)

Is 1 a factor of 12? (Yes, the reason is because $1 \times 12 = 12$.)

Is 1 a factor of 9? (Yes, the reason is because $1 \times 9 = 9$.)

Is 1 a factor of any, all, or some whole numbers? (It is a factor of all whole numbers. $1 \times 0 = 0$, $1 \times 5 = 5$, etc., or $1 \times n = n$.)

If a number, n, is multiplied by 1, what is the result? (n)

Write this relationship as a mathematical sentence. ($n \times 1 = n$)

If a number, n, is multiplied by 0, what is the result? (0) Write this relationship as a mathematical sentence. ($n \times 0 = 0$)

You have agreed that 1 is a factor of any whole number. If so, any whole number may be divided by one to find an unknown factor. Give some examples. ($12 \div 1 = 12$; $5 \div 1 = 5$; $925 \div 1 = 925$.)

You have also found that 0 is not a factor of any whole number except zero. Because 0 is not a factor of any other whole number, we cannot divide any other whole number by zero.

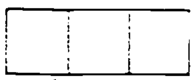
> With stronger pupils, you may find it profitable to discuss the following questions.
>
> What number can be substituted for n so that $0 \times n = 0$ is a true statement? (any number)
>
> Is there any number which can be substituted for n to make $0 \times n = 1$ a true statement? (no number)

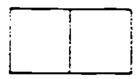
ONE OR ZERO AS A FACTOR

Exercise Set 16

1. What number is a member of every set of factors? Why?
(1 because $1 \times n = n$)

2. Write a mathematical sentence suggested by each array shown below.

a. 
($1 \times 3 = 3$)

b. 
($1 \times 2 = 2$)

c. 
($1 \times 6 = 6$)

3. Find the unknown factor.

   a. $1 \times n = 9$ $\quad n = 9$

   b. $n \times 4 = 4$ $\quad n = 1$

   c. $1 \times n = 53$ $\quad n = 53$

   d. $72 \times n = 72$ $\quad n = 1$

   e. $923 \times n = 923$ $\quad n = 1$

4. Write the multiplication sentence belonging to a 1 by 287 array. ($1 \times 287 = 287$)

5. Write two of the factors of 2,877. (1 and 2,877)

6. Complete the following sentences.

   Example: $2 \times 5 = 10$

   a. $1 \times 5 = 5$

   b. $0 \times 5 = 0$

   c. $3 \times 2 = 6$

   d. $3 \times 1 = 3$

   e. $3 \times 0 = 0$

   f. $1 \times 0 = 0$

   g. $0 \times 2 = 0$

   h. $2 \times 0 = 0$

7. a. What is $0 \times 9{,}726 = ?$ (0)

   b. What product do you always get when one of the factors is zero? Why? (0, because 0 times any number is 0)

   c. What numbers are factors of 0? (All numbers)

   d. Is 0 a factor of 0? Why? (Yes, $0 \times 1 = 0$ is one example which shows this.)

8. How many arrays have 11 elements? (Just two arrays have 11 elements — a 1 by 11 array and an 11 by 1 array. This is because 1 and 11 are the only factors of 11.)

CLOSURE

Objective: To help children understand that the operation of multiplication is always possible within the set of whole numbers, and that the operation of division is not always possible within the set of whole numbers

> Within the set of whole numbers any two numbers we choose to multiply give us a whole number as a product. This is not always true if we use the operation of division. If we choose 6 and 3, we get the whole number 2; but if we choose 9 and 2, we do not get a whole number; and if we choose 5 and 0, we do not get any number.
>
> In describing the property of closure, we say the set of whole numbers is closed under the operation of multiplication; the set of whole numbers is not closed under division.

Exploration:

What is the smallest whole number you know? (0) What is the largest? (999, 1,000,000; etc.) How many whole numbers are there? (There are more than I can count.)

When you choose any two of these whole numbers, can you add them? (Yes) Try some. ($7 + 9 = 16$; $25 + 62 = 87$, etc.) Is it always possible to add two whole numbers? (Yes) Is it always possible to subtract two whole numbers? (No) Try some. ($7 - 2 = 5$; $2 - 7 = n$; $5 - 5 = 0$; $1 - 8 = n$)

Is it always possible to multiply two whole numbers? (Yes) How do you know? (We can imagine an array with any number of rows and columns.)

We describe this by saying that the set of whole numbers is <u>closed</u> under multiplication. This just means that the product of two whole numbers is a whole number.

Is it always possible to divide two whole numbers? (No) I will choose some pairs for you and put them in a chart. If you get a whole number, write this number in the unknown factor column. If you do not, write "no whole number."

	Division Pair	Unknown Factor
(a)	12 , 3	4
(b)	3 , 12	no whole number
(c)	15 , 5	3
(d)	5 , 15	no whole number
(e)	0 , 3	0
(f)	3 , 0	no whole number

Could you divide each pair? (No) Explain why you could not and get a whole number for a result. (3 cannot be divided by 12 because 12 is not a factor of 3, etc.)

Is it always possible to divide any pair of whole numbers? (No) Give other examples.

Is the set of whole numbers closed under division? (No, because it is not always possible to divide and get a whole number answer.)

MULTIPLYING AND DIVIDING WHOLE NUMBERS

If two whole numbers are multiplied, we always get another whole number. We cannot always divide two whole numbers if we want a whole number answer.

There is not always a whole number to use as an unknown factor. For example,

$$[illegible] = 3 \times n$$

and

$$[illegible] = n \times 3$$

no matter what whole number n names.

It is always possible to multiply two whole numbers because there is always a third whole number to use as a product. This means the set of whole numbers is <u>closed</u> under multiplication.

It is not always possible to divide two whole numbers because there is not always a third whole number to use as an unknown factor. For example, there is no whole number n so that $10 \div 3 = n$. This means that the set of whole numbers is <u>not</u> <u>closed</u> under division.

Exercise Set 17

1. If possible, complete each mathematical sentence so it will be a true sentence. If there is no whole number to use as a result, answer "no".

   Example: [illegible]

   a. 15 ÷ 15 = [illegible]
   b. 5 − 0 = 5
   c. 6 ÷ 24 = no
   d. 7,255 × 0 = 0
   e. 15 − 25 = no
   f. 5 ÷ 5 = [illegible]
   g. 6 × 7 = 42

2. Complete the following statements by using either "always" or "not always." Give an example from exercise 1.

   a. Within the set of whole numbers, addition is always possible.
   b. Within the set of whole numbers, multiplication is always possible.
   c. Within the set of whole numbers, subtraction is not always possible. Give an example. 15 − 25 ≠ a whole number
   d. Within the set of whole numbers, division is not always possible. Give an example. 6 ÷ 24 ≠ a whole number.

3. Using what you discovered in 1 and 2 complete the following statements by using either "closed" or "not closed."

   a. Within the set of whole numbers, multiplication is closed.
   b. Within the set of whole numbers, division is not closed.
   c. Within the set of whole numbers, subtraction is not closed.
   d. Within the set of whole numbers, addition is closed.

DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

Objective: To help children understand and use the property of multiplication illustrated by

$$7 \times (10 + 3) = (7 \times 10) + (7 \times 3),$$

and stated by the formula

$$a \times (b + c) = (a \times b) + (a \times c).$$

Materials: A 7 by 13 array and a 9 by 14 array

Vocabulary: Distributive property of multiplication over addition

Before introducing the distributive property of multiplication over addition you should review the structure of the decimal system, emphasizing two-digit numerals. Give special attention to multiples of 10, e.g., 2×10, 3×10, etc.

The distributive property is of very great practical importance. It is tacitly used whenever we find a product like 23×3 by the usual algorism of multiplication.

Usually we are not given a factor as a sum, but we may write it as a sum in several ways. For example, consider the multiplication problem $7 \times 13 = n$. We may rename 13 as $12 + 1$, $11 + 2$, $10 + 3$, etc. The distributive property then gives us

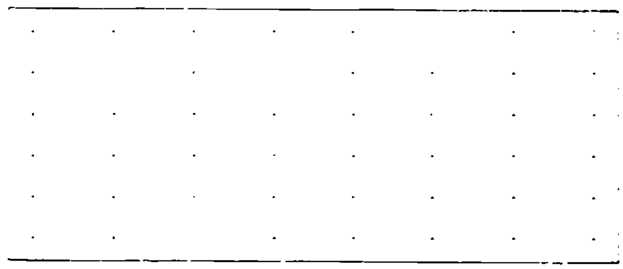
$$\begin{aligned} 7 \times 13 &= 7 \times (12 + 1) \\ &= (7 \times 12) + (7 \times 1) \\ 7 \times 13 &= 7 \times (11 + 2) \\ &= (7 \times 11) + (7 \times 2) \\ 7 \times 13 &= 7 \times (10 + 3) \\ &= (7 \times 10) + (7 \times 3), \text{ etc.} \end{aligned}$$

Each renaming of 13 renames the product 7×13. It is this fact that makes the distributive property so useful. Complicated multiplications may thus be reduced to known multiplication facts and addition.

The children will give many names for the factor to be renamed as a sum. In working with children, guide them to see that usually one form of renaming is most convenient to use. For example, $6 + 7$, $8 + 5$, $9 + 4$, $10 + 3$, $11 + 2$, and $12 + 1$ are all names for 13; but $10 + 3$ is the most convenient to use.

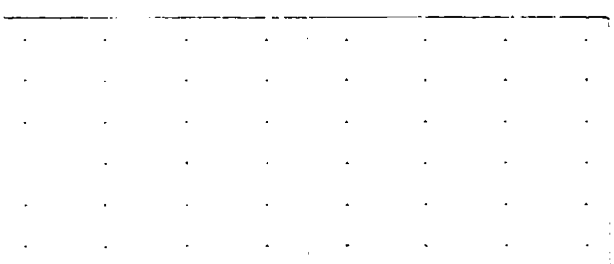
Exploration:

Let us examine a 6 by 8 array. By using the array below we can discover a method for finding a product we do not know.

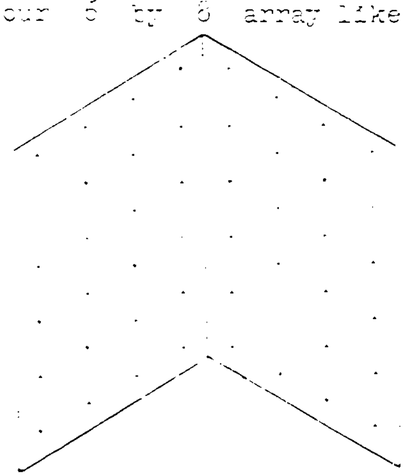


If you do not know $6 \times 8 = 48$ but you do know 6×1, 6×2, ... , 6×7, how can you find 6×8 as you look at the array? (We can think of the array separated into two equal parts. Each will have 6 rows of   columns. Then $6 \times 8 = 24 + 24 = 48$.)

When we think this way, we really are separating the 6 by 8 array into two arrays. I will indicate them on the chalkboard.



We can indicate the separation of an array into two arrays by simply folding our 6 by 8 array like this.



[illegible] draw a line to indicate where we are separating them.

State a mathematical sentence giving a relationship between the number of <u>columns</u> of the 6 by 8 array and the number of <u>columns</u> of the two 6 by 4 arrays. (8 = 4 + 4)

I shall write this sentence on the chalkboard so we may refer to it later.

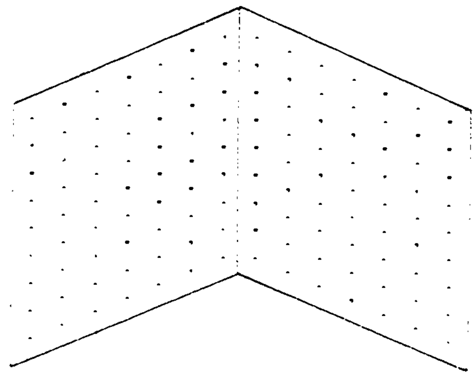
Could we separate the 6 by 8 array into two other arrays in a different way? (Yes, we could separate it into a 6 by 2 array and a 6 by 6 array.) I will indicate this separation [illegible] the array. How do we find the product 6 × 8 with this separation? (6 × 8 = 6 × (2 + 6), 6 × 8 = (6 × 2) + (6 × 6), 6 × 8 = 12 + 36 = 48.)

What is the relationship between the number of columns in the 6 by 8 array and the number of columns in the two arrays into which we have separated this time? (8 = 2 + 6)

I will write this sentence on the chalkboard under 8 = 4 + 4.

> The children may wish to separate the array in other ways to show that 6 × 8 = 48; [illegible] 6 × 8 = 6 × (3 + 5), 6 × 8 = (6 × 3) + (6 × 5). [illegible] on the chalkboard (for reference) [illegible] the relationship between the number of columns in the 6 by 8 array and the number of columns in the two arrays into which it was separated.
>
> Ask the children if the 6 by 8 array could have been separated into more than two parts. This discussion should lead to statements like these: 6 × 8 = 6 × (2 + 2 + 4), (6 × 2) + (6 × 2) + (6 × 4) = 12 + 12 + 24 = 48.

[illegible] is part of an array for which you do not know the number of [illegible]. It has [illegible] rows and 14 columns. Can you separate this array into two arrays so that you know the number of [illegible] in each array? Indicate the separation by [illegible].



(Yes, I can make two arrays each 9 by 7. Each array will contain 63 dots. 9 × 14 = (9 × 7) + (9 × 7). 9 × 14 = 63 + 63 = 126.)

Can anyone describe the array in another way?

> The teacher should follow the discussion of finding the product of 9 × 14, first by summarizing the ways 14 columns may be separated: 14 = 7 + 7; 14 = 8 + 6; 14 = 9 + 5; 14 = 10 + 4. Then summarize the ways the product of 9 and 14 may be discovered.

9 × 14 = 9 × (7 + 7) = (9 × 7) + (9 × 7) = 63 + 63 = 126
9 × 14 = 9 × (9 + 5) = (9 × 9) + (9 × 5) = 81 + 45 = 126
9 × 14 = 9 × (8 + 6) = (9 × 8) + (9 × 6) = 72 + 54 = 126
9 × 14 = 9 × (11 + 3) = (9 × 11) + (9 × 3) = 99 + 27 = 126

> In the examples above and in similar examples, it is understood that all multiplication should be done before the addition.

Let us try to describe in mathematical language how we found the products 9 × 8 and 9 × 14. If we can do this, we will know how to find other products without spending this time making arrays and separating them to represent the products.

Let us first think of the product, 9 × 8. Look at the mathematical sentences 8 = 4 + 4, 8 = 2 + 6, 8 = 3 + 5 on the chalkboard. Each sentence renames the number of columns of the array. For each renaming, we found a way to find the

product 6 × 8. Tell me those ways again. (6 × 8 = 6 × (4 + 4) = (6 × 4) + (6 × 4) = 24 + 24 = 48, etc.) Can you think of any other ways we could rename 8 ? (8 = 7 + 1; 8 = 2 + 2 + 4; 8 = 10 − 2.)

The property of multiplication we have been discussing is called the distributive property of multiplication over addition. To make it clearer, I will write what we have found like this:

(a) 6 × (4 + 4) = (6 × 4) + (6 × 4)
(b) 6 × (2 + 6) = (6 × 2) + (6 × 6)
(c) 6 × (3 + 5) = (6 × 3) + (6 × 5)

The distributive property is very useful in using known products to find unknown products. We can illustrate this with the product 7 × 14.

> The children should follow a similar exploration through the examination of arrays which are 8 by 13, 8 by 15, etc. The arrays should be folded and then the corresponding mathematical sentence should be written; e.g., 8 × 13 = (8 × 7) + (8 × 6) = 56 + 48 = 104. On the chalkboard should be written a column of statements of ways of finding each product. One column would look like the one of page 321 for 7 × 14. Another would look like this one for finding 8 × 13.
>
> 8 × 13 = 8 × (4 + 9) = (8 × 4) + (8 × 9) = 32 + 72 = 104
> 8 × 13 = 8 × (5 + 8) = (8 × 5) + (8 × 8) = 40 + 64 = 104
> 8 × 13 = 8 × (6 + 7) = (8 × 6) + (8 × 7) = 48 + 56 = 104
> 8 × 13 = 8 × (10 + 3) = (8 × 10) + (8 × 3) = 80 + 24 = 104

We have now shown several ways of finding a product. They are all illustrations of the distributive property of multiplication over addition.

Let us review the way you renamed 13 to find 8 × 13. It was renamed 4 + 9, 5 + 8, 6 + 7, and 10 + 3.

How could you rename 25 in 7 × 25? Remember it must be renamed so you know the multiplication facts required for finding the product 7 × 25. (25 could be renamed as 8 + 8 + 9, 10 + 10 + 5.)

If we need to find a product quickly, it is wise for us to have a plan for renaming a factor. Look at all the factors which you have renamed and try to decide on a convenient way to rename a factor.

Which was the most convenient way to rename 25 in 7 × 25? ([illegible] + [illegible])

Which was the most convenient way to rename 16 in 8 × 16? (10 + 6)

Can you tell what seems to be the most convenient way to rename a factor to help us find the product? (It seems to be most convenient to rename it in tens and ones.)

How would you conveniently rename [illegible] to find [illegible] × [illegible]? ([illegible] or [illegible]) Find [illegible] × [illegible].

How would you conveniently rename 215 to find [illegible] × 215? ([illegible] or (200 + 10 + 5)) Find [illegible] × 215.

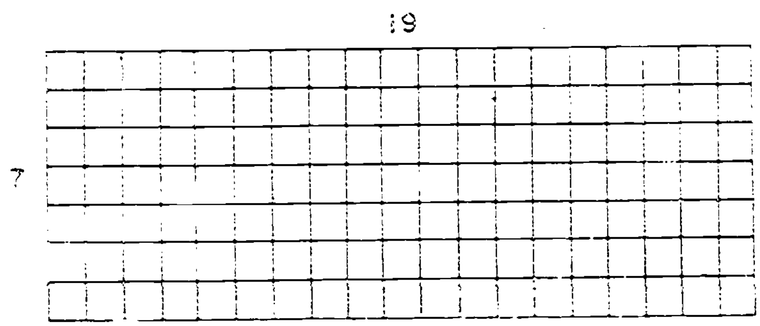
We shall use the distributive property of multiplication [illegible] more and more as we learn to multiply numbers [illegible] have many digits.

> So far we have used the distributive property [illegible] as stated at the beginning of this [illegible]; i.e., the factor on the right has been renamed as a sum. This corresponds to the separation of an array by columns. While it is simpler at first to rename only factors on the right and to separate arrays only by columns, there is another form of the distributive property; namely
>
> $(a + b) \times c = (a \times c) + (b \times c)$.
>
> For example,
>
> [illegible] × [illegible] = ([illegible] + [illegible]) × [illegible]
> = ([illegible] × [illegible]) + ([illegible] × [illegible]).
>
> [illegible] this property by renaming the left factor. It corresponds to the separation of an [illegible] by [illegible] array by rows.
>
> The teacher probably can get the class to tell which factor should be renamed in examples [illegible] × 16. It is important for pupils to realize that the choice is a matter of convenience. Some may point out that, since [illegible] × [illegible], the form of the distributive property is [illegible].

THE DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

Working Together

Make an array like this one on a piece of paper you can fold.

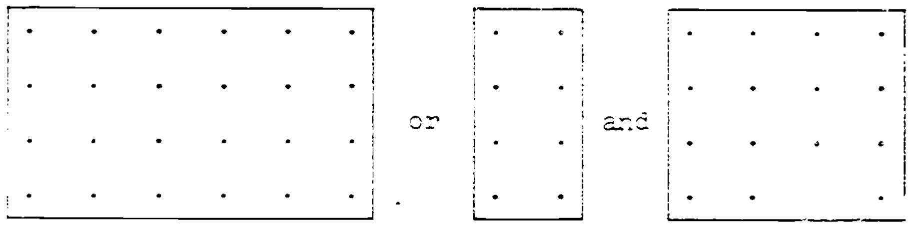


1. By folding the array you can separate the 19 columns of the array into two parts. This can be done in 18 different ways. Write a mathematical sentence for each fold you make.

   Example: 19 = 7 + 12

   19 = 6 + 13, etc.

2. Here is one array separated into two smaller arrays.



$(n = 4 \times 6)$ Array A $(p = 4 \times 2)$ Array B $(q = 4 \times 4)$ Array C

a. n is the number of dots in Array A. How many dots are there in Array A? 24 dots

b. p is the number of dots in Array B. How many dots are there in Array B? 8 dots

c. q is the number of dots in Array C. How many dots are there in Array C? 16 dots

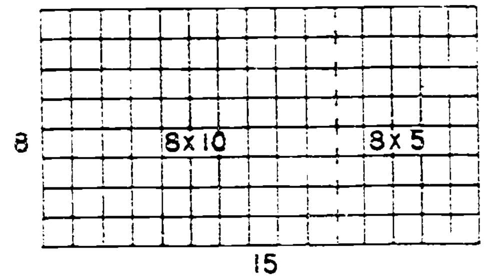
d. When you add the number of elements in Array B and Array C together, is the result the same as n? Yes

e. Does $n = p + q$? Yes

f. Does $24 = 8 + 16$? Yes

g. Does $4 \times 6 = (4 \times 2) + (4 \times 4)$? Yes

3.



$$8 \times 15 = (8 \times 10) + (8 \times 5)$$

The dotted line shows a possible way to fold the array. The sentence below the picture shows the relation between the whole array and the two smaller arrays which the fold makes. The array can be folded in many other ways. Find 6 different ways of separating the array. Write the mathematical sentence for each separation.

$8 \times 15 = (8 \times 2) + (8 \times 13)$
$8 \times 15 = (8 \times 3) + (8 \times 12)$
$8 \times 15 = (8 \times 4) + (8 \times 11)$
etc.

4. Make an array from squared paper to show the product of 9×13. Find 6 different ways of separating the array. Write the mathematical sentence for each separation.

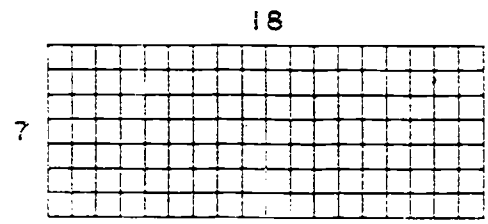
$9 \times 13 = (9 \times 10) + (9 \times 3)$
$9 \times 13 = (9 \times 11) + (9 \times 2)$
etc.

5. Which of the sentences you wrote is the easiest one for you to use to find the product of 8 and 15? Answers will vary. Provided they can be justified, they should be accepted; however, children should be led to realize that for most people, renaming 15 as 10 + 5 is the easiest.

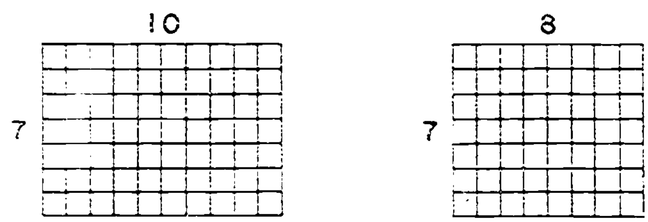
6. Which of the sentences you wrote is the easiest one for you to use to find the product of 9 and 13? Answers will vary as in Exercise 5. $9 \times 13 = (9 \times 10) + (9 \times 3)$ is probably the easiest.

Summary

To find the product of 7 and 18, think of a 7 by 18 array.



Separate it into two arrays showing products you already know. For example,



Find the products separately and add them to get the total number of elements in the 7 by 18 array.

$$\begin{aligned} 7 \times 18 &= 7 \times (10 + 8) \\ &= (7 \times 10) + (7 \times 8) \\ &= \quad 70 \quad + \quad 56 \\ &= \quad 126 \end{aligned}$$

When we write $(7 \times 10) + (7 \times 8)$ in place of 7×18 we are using the distributive property of multiplication over addition.

USING THE DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

Exercise Set 1c

Rename the second factor of each of the following so it is convenient for you to multiply. Use the distributive property of multiplication over addition to find the product. Study the example before you begin.

Example: $4 \times 17 = 4 \times (10 + 7)$

$= (4 \times 10) + (4 \times 7)$

$= 40 + 28$

$= 68$

1. $3 \times 15 = 3 \times (10 + 5) = (3 \times 10) + (3 \times 5) = 30 + 15 = 45$

2. $7 \times 12 = 7 \times (10 + 2) = (7 \times 10) + (7 \times 2) = 70 + 14 = 84$

3. $5 \times 16 = 5 \times (10 + 6) = (5 \times 10) + (5 \times 6) = 50 + 30 = 80$

4. $6 \times 19 = 6 \times (10 + 9) = (6 \times 10) + (6 \times 9) = 60 + 54 = 114$

5. $8 \times 13 = 8 \times (10 + 3) = (8 \times 10) + (8 \times 3) = 80 + 24 = 104$

Example: $7 \times 26 = 7 \times (20 + 6)$

$= (7 \times 10) + (7 \times 10) + (7 \times 6)$

$= 70 + 70 + 42$

$= 182$

6. $7 \times 26 = 7 \times (20 + 6) = (7 \times 20) + (\underline{7} \times \underline{6}) = 140 + \underline{42} = \underline{182}$

7. $5 \times 34 = 5 \times (30 + 4) = (5 \times 30) + (5 \times 4) = 150 + 20 = 170$

8. $6 \times 28 = 6 \times (20 + 8) = (6 \times 20) + (6 \times 8) = 120 + 48 = 168$

9. $9 \times 22 = 9 \times (20 + 2) = (9 \times 20) + (9 \times 2) = 180 + 18 = 198$

Example: $4 \times 153 = 4 \times (100 + 50 + 3)$

$= (4 \times 100) + (4 \times 50) + (4 \times 3)$

$= 400 + 200 + 12$

$= 612$

10. $3 \times 162 = 3 \times (100 + 60 + 2) = (3 \times 100) + (3 \times 60) + (3 \times 2) =$
$300 + 180 + 6 = 486$
11. $4 \times 147 = 4 \times (100 + 40 + 7) = (4 \times 100) + (4 \times 40) + (4 \times 7) = 400 + 160 + 28 = 588$
12. $5 \times 112 = 5 \times (100 + 10 + 2) = (5 \times 100) + (5 \times 10) + (5 \times 2) = 500 + 50 + 10 = 560$
13. $7 \times 120 = 7 \times (100 + 20) = (7 \times 100) + (7 \times 20) = 700 + 140 = 840$
14. $6 \times 132 = 6 \times (100 + 30 + 2) = (6 \times 100) + (6 \times 30) + (6 \times 2) = 600 + 180 + 12 = 792$
15. $9 \times 111 = 9 \times (100 + 10 + 1) = (9 \times 100) + (9 \times 10) + (9 \times 1) = 900 + 90 + 9 = 999$
16. $3 \times 243 = 3 \times (200 + 40 + 3) = (3 \times 200) + (3 \times 40) + (3 \times 3) = 600 + 120 + 9 = 729$
17. $2 \times 361 = 2 \times (300 + 60 + 1) = (2 \times 300) + (2 \times 60) + (2 \times 1) = 600 + 120 + 2 = 722$

DISTRIBUTIVE PROPERTY OF DIVISION OVER ADDITION

Objective: To help children understand and use the property of division exemplified by:

$$(40 + 8) \div 4 = (40 \div 4) + (8 \div 4)$$

and stated by the sentence:

$$(b + c) \div a = (b \div a) + (c \div a)$$

Materials: [illegible] array

Exploration:

The distributive property of division over addition is fully as important as the distributive property of multiplication over addition. It is the basis for the usual "long division" algorism. If the children understand the use of this property in simple problems, the long division algorism will be easier to learn.

The distributive property of division over addition is a direct consequence of the distributive property of multiplication over addition. Just as division is a way of looking at multiplication, the distributive property of division over addition is a way of looking at the distributive property of multiplication over addition. In a typical situation we find $48 \div 4$ by separating a 4 row, 48 element array into two 4 row arrays with known numbers of elements; e.g., a 40 element array and an 8 element array.

We then observe that the number of columns in array A is the sum of the number of columns in arrays B and C. In the language of division, this means

Number of Columns in Array A		Number of Columns in Array B		Number of Columns in Array C
$48 \div 4$	$= (40 + 8) \div 4 =$	$(40 \div 4)$	$+$	$(8 \div 4)$

In the language of multiplication this says $4 \times$ (number of columns in A) $= 4 \times$ (number of columns in B) $+ 4 \times$ (number of columns in C).

The diagram on page 331 and the suggestions in the explanation following should help to illustrate the basic idea. It is important for the teacher to observe that the process of separating a 4 row, 48 element array corresponds to the writing of 48 as a sum of numbers each with 4 as a factor, and one of which is a multiple of 10.

There are many interesting things we can find about division by studying arrays and mathematical sentences which indicate division.

Let us study an array and the mathematical sentences which describe it.

```
. . . . . . . . . . . . .
. . . . . . . . . . . . .     3 × n = 39
                                  or
. . . . . . . . . . . . .     n = 39 ÷ 3
```

Most of you can find n, in $n = 39 \div 3$. What is it? (13)

What is n in this sentence, $n = 144 \div 3$? (We don't know.) What is n in this sentence, $5225 \div 55$? (We don't know.) Most of you cannot find n. You will soon be able to, but first we must find a way. That is why we are studying the array and the mathematical sentence, $n = 39 \div 3$.

Do you know a multiplication fact to help you? (No) It is not one of our basic facts.

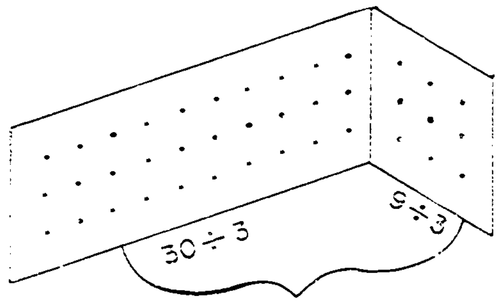
Let us use the same method here that we did when we tried to find a product we didn't know. We will use part of the array at a time. We will fold it into parts so that the number of dots in each part can be divided by 3.

Into what parts can you separate an array of 39 so you can divide each part by 3? (30 + 9, 15 + 15 + 9, 18 + 18 + 3, 21 + 18, 12 + 12 + 12 + 3, etc.)

> The children may fold the array and write many more names for 39.

We have listed many of the ways the array can be folded so that you can divide each part by 3. Let us examine a few of them.

Fold the array into parts of 30 and 9. What is $30 \div 3$? (10) What is $9 \div 3$? (3) If 10 and 3 are added, the sum of the two factors is 13. What does 13 represent in $n = 39 \div 3$? (The unknown factor)



$$n = 39 \div 3 = (30 + 9) \div 3$$
$$= (30 \div 3) + (9 \div 3)$$
$$= 10 + 3$$
$$= 13$$

The next way of folding you suggested was to make parts of 15, 15, and 9. Fold it and show the division.

> The children under the guidance of the teacher should make the folds and show this type of work on the chalkboard.
>
> $n = 39 \div 3$
> $= (15 + 15 + 9) \div 3$
> $= (15 \div 3) + (15 \div 3) + (9 \div 3)$
> $= 5 + 5 + 3$
> $= 13$

Suppose you don't have an array to fold. Can you tell me another name we can use for 48 to find n in $n = 48 \div 4$? (Yes, $24 + 24$; $12 + 12 + 24$; $40 + 8$; etc.)

> The children should find n in $48 \div 4$ by using several of these names for 48 and dividing each part of 48 by 4 to find $48 \div 4$.

This seems to be a good way for us to divide when we do not know a multiplication fact to use. What did you do? (We wrote the product as a sum. We divided each addend and then added the results.)

In [illegible] ÷ [illegible], what was the easiest way to rename [illegible] in order to divide it by [illegible]? (60 + 9)

In [illegible] ÷ [illegible], what was the easiest way to rename [illegible] in order to divide it by [illegible]? ([illegible] + 8)

In [illegible] ÷ [illegible], what is the easiest way to rename [illegible] in order to divide it by [illegible]? ([illegible] + [illegible] + [illegible])

What is the easiest way to think about each of these products in order to divide it [illegible]? (The easiest way is to think about the product in hundreds, tens, and ones.)

[illegible] ÷ [illegible] we think of [illegible] as [illegible]00 + 60 [illegible] and divide each of the numbers [illegible], [illegible]0, and [illegible] by 3.

[illegible]

[illegible] = ([illegible] ÷ [illegible]) + ([illegible]0 ÷ 3) + ([illegible] ÷ [illegible])

[illegible] + [illegible]20 + [illegible]

[illegible]

[illegible] examples which illustrate the distributive property of division over addition.

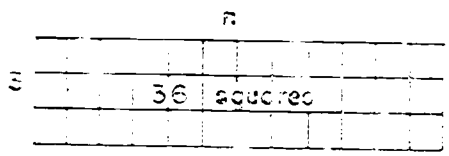
> [illegible] is [illegible] question that may arise since [illegible] are comparing the distributive property of multiplication over addition and the distributive property of division over addition. We know that [illegible] × ([illegible]) = ([illegible] + [illegible]) × [illegible] because the commutative property holds for multiplication. However, since the commutative property does not hold for division, [illegible] ([illegible] + [illegible]) ÷ [illegible] ≠ [illegible] ÷ ([illegible] + [illegible]). [illegible] the distributive property of division over addition, the following form is the only form to [illegible]:
>
> ([illegible] + [illegible]) ÷ [illegible]

[illegible]

USING THE DISTRIBUTIVE PROPERTY OF DIVISION OVER ADDITION

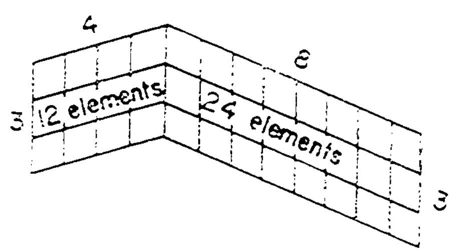
[illegible]

1. Make an array of 36 squares. Use a piece of paper which may be folded. Make your array like the one below.



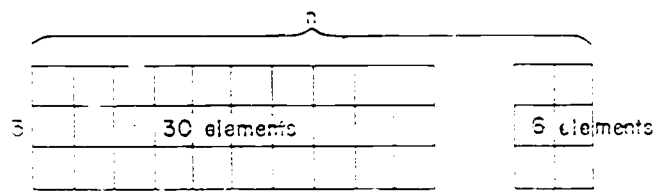
2. By folding the array, find 5 ways you can separate the 36 squares into two parts so that the number of squares in each part may be divided by 3. Write a mathematical sentence to show what you have done.

Example:



36 = 12 + 24

2. Suppose that you separated the array in Exercise 1 in this way.



$$36 = 30 + 6$$

We then can think:

$36 \div 3 = n$

$36 \div 3 = (30 + 6) \div 3$

$36 \div 3 = (30 \div 3) + (6 \div 3)$

$36 \div 3 = 10 + 2$

$36 \div 3 = 12$

3. Select another way in which you separated the 36-element array in Exercise 1. Show as we did above in Exercise 2 that $36 \div 3 = 12$ when using this other separation.

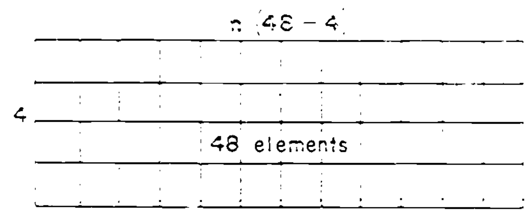
4. Do you think that one way of separating the array makes it easier to find $36 \div 3$ than other ways of separating the array? Why?

Summary

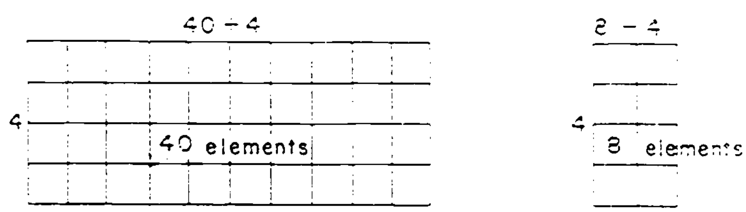
To find an unknown factor n as in

$$48 = 4 \times n$$

we can think of an array with 48 elements in 4 rows and n columns. If $48 = 4 \times n$, then $n = 48 \div 4$.



Think of separating this array into two arrays, so that the number of elements in each part can be divided easily by 4; for example:



Find the number of columns in each array. Add these numbers to find the number of columns in the 48 element array.

$$\begin{aligned} 48 \div 4 &= (40 + 8) \div 4 \\ &= (40 \div 4) + (8 \div 4) \\ &= 10 + 2 \\ &= 12 \end{aligned}$$

When we write $(40 \div 4) + (8 \div 4)$ in place of $48 \div 4$, we are using the distributive property of division over addition.

[illegible] WITH THE DISTRIBUTIVE PROPERTY OF DIVISION OVER ADDITION

[illegible] Set [illegible]

[illegible] distributive property of division over addition, find the [illegible].

Example: [illegible] ÷ [illegible] = (20 + [illegible]) ÷ [illegible]

= (20 ÷ [illegible]) + ([illegible] ÷ [illegible])

= 10 + [illegible]

= 1[illegible]

1. [illegible] ÷ 3 = (60 + 6) ÷ 3 = (60 ÷ 3) + (6 ÷ 3) = 20 + 2 = 22

2. [illegible] ÷ [illegible] = (80 + 4) ÷ [illegible] = (80 ÷ [illegible]) + ([illegible] ÷ [illegible]) = 2[illegible] + [illegible] = 2[illegible]

3. [illegible] ÷ [illegible] = ([illegible] + [illegible]) ÷ 2 = ([illegible] ÷ 2) + ([illegible] ÷ 2) = 3[illegible] + [illegible] = 3[illegible]

4. [illegible] ÷ 3 = (90 + 6) ÷ 3 = (90 ÷ 3) + (6 ÷ 3) = 30 + 2 = 32

5. [illegible] ÷ 4 = (80 + 8) ÷ 4 = (80 ÷ 4) + (8 ÷ 4) = 20 + 2 = 22

6. [illegible] ÷ 2 = (80 + 6) ÷ 2 = (80 ÷ 2) + (6 ÷ 2) = 40 + 3 = 43

7. [illegible] ÷ 3 = (60 + 9) ÷ 3 = (60 ÷ 3) + (9 ÷ 3) = 20 + 3 = 23

BRAINTWISTER: Study the examples before trying exercises 8 through 12.

Example: $284 \div 2 = (200 + 80 + 4) \div 2$

$= (200 \div 2) + (80 \div 2) + (4 \div 2)$

$= 100 + 40 + 2$

$= 142$

Example: $369 \div 3 = (300 + 60 + 9) \div 3$

$= (300 \div 3) + (60 \div 3) + (9 \div 3)$

$= 100 + 20 + 3$

$= 123$

8. $468 \div 2 = (400 + 60 + 8) \div 2 = (400 \div 2) + (60 \div 2) + (8 \div 2) =$
$200 + 30 + 4 = 234$

9. $999 \div 9 = (900 + 90 + 9) \div 9 = (900 \div 9) + (90 \div 9) + (9 \div 9) =$
$100 + 10 + 1 = 111$

10. $693 \div 3 = (600 + 90 + 3) \div 3 = (600 \div 3) + (90 \div 3) + (3 \div 3) =$
$200 + 30 + 1 = 231$

11. $840 \div 8 = (800 + 40) \div 8 = (800 \div 8) + (40 \div 8) = 100 + 5 = 105$

12. $75 \div 3 = (60 + 15) \div 3 = (60 \div 3) + (15 \div 3) = 20 + 5 = 25$

THE ASSOCIATIVE PROPERTY OF MULTIPLICATION

Objective: To help children understand

(1) that the product of three factors a, b, and c may be shown by either $(a \times b) \times c$ or $a \times (b \times c)$

(2) that we can write $a \times b \times c$ without ambiguity because $(a \times b) \times c = a \times (b \times c)$

Exploration:

> It should be emphasized that multiplication is a binary operation. In other words, just two numbers can be multiplied in one multiplication operation. Consequently in multiplying three numbers, the multiplication operation must be performed twice. This may be done by grouping, or associating, the numbers to be multiplied. For example, to find $3 \times 4 \times 5$, we may find $(3 \times 4) \times 5$ or $3 \times (4 \times 5)$. In this example, the teacher should be sure to indicate the partial products 3×4 and 4×5 so that the children will understand one multiplication is 12×5, and the other multiplication is 3×20. Both products, of course, are renamed 60.
>
> The teacher should be careful not to change the order of the factors when explaining the associative property to the children. Write
>
> $(3 \times 2) \times 4 = 3 \times (2 \times 4)$, $6 \times 4 = 3 \times 8$
>
> not $(3 \times 2) \times 4 = 2 \times (3 \times 4)$, $6 \times 4 = 2 \times 12$.
>
> The latter form is, of course, correct; but it uses the commutative property as well as the associative property of multiplication.

We have been discovering properties of multiplication. We have found that multiplication, like addition, has a commutative property. A question we should consider is "Does multiplication, like addition, also have an associative property?"

Let us first review the associative property of addition. Illustrate it with these addends: 3, 4, 5. $((3 + 4) + 5 = 3 + (4 + 5))$ Illustrate the property with several numbers a, b, and c. $((a + b) + c = a + (b + c))$

In the sentence $(a + b) + c = a + (b + c)$ there are three addends, a, b, and c to be added. This property was very important for us to know when we were learning how to add large numbers.

Write the sentence which shows the grouping of three addends. In each place where there is a symbol showing addition we will put a symbol showing multiplication. $(a \times b) \times c = a \times (b \times c)$

We must test this statement by replacing the pictures a, b, and c by whole numbers. Write the sentence with $a = 3$, $b = 3$, and $c = 2$. $(3 \times 3) \times 2 = 3 \times (3 \times 2)$.

Is this a true statement? (Yes, $9 \times 2 = 18$, $3 \times 6 = 18$, $9 \times 2 = 3 \times 6$)

> Continue to substitute other numbers for a, b, and c to test the truth of the mathematical sentence
> $(a \times b) \times c = a \times (b \times c)$.

Does the associative property seem to apply in multiplication? (Yes, it did in all the sentences we tested.) It is a very important property of multiplication.

We have used many illustrations of the associative property of multiplication. We must now decide if our original sentence showing grouping is a correct one. Here is the sentence:

$(a \times b) \times c = a \times (b \times c)$. (Yes, we have tested it. a, b, and c may be any whole number we choose.)

Now suppose I write

$$3 \times 3 \times 2 .$$

Is there any doubt about what number this names? (No, someone might think it means $3 \times (3 \times 2)$. Someone else might think it means $(3 \times 3) \times 2$, but these are both names for 18.) How can I find $3 \times 3 \times 2$? (You can find it either by finding $3 \times 3 = 9$ and then $9 \times 2 = 18$, or by finding $3 \times 2 = 6$ and then $3 \times 6 = 18$.) From now on we will leave out parentheses unless we have a reason to show a particular grouping.

You have used numbers less than 10 to illustrate. Could the associative property be applied to larger numbers? (Yes) We shall work one or two exercises together.

Do you know these products: 1 × 10, 2 × 10, ... 10 × 10, ... 1 × 100, 2 × 100, ... 9 × 100 ?

If you know these, we can find n = 3 × 20 .

3 × 20 = 3 × (2 × 10)	Rename 20 as (2 × 10).
= (3 × 2) × 10	Use the associative property.
= 6 × 10	
= 60	

Another example where both the associative property and commutative property of multiplication are used is 30 × 20 = n.

n = 30 × 20	
= (3 × 10) × (2 × 10)	Rename 30 and 20.
= [(3 × 10) × 2] × 10	Use the associative property.
= [3 × (10 × 2)] × 10	Use the associative property.
= [3 × (2 × 10)] × 10	Use the commutative property.
= (3 × 2) × (10 × 10)	Use the associative property.
= 6 × 100	
= 600	

You will want to supplement with additional examples such as 3 × 20, 30 × 60, etc.

Pupils may need to have brackets explained. We use brackets in situations in which one set of parentheses is needed inside another. Brackets are used rather than another pair of parentheses for the sake of clarity.

> The children should have an opportunity to summarize with specific numbers, and with general numbers (a, b, c, etc.) each of the following.
> 1. Relation of multiplication and division
> 2. Zero and one as factors and products
> 3. Closure property
> 4. Distributive property of division
> 5. Commutative property
> 6. Associative property of multiplication
> 7. Distributive property of multiplication

THE ASSOCIATIVE PROPERTY OF MULTIPLICATION

Working Together

1. Suppose you were told to multiply 2, 3, and 4.

$$2 \times 3 \times 4 = n$$

(a) Can you multiply the three numbers at the same time?

(b) Find the product when you multiply 2×3, and then multiply the result by 4.

(c) Find the product when you multiply 3 and 4 and then multiply the result by 2.

(d) Are the final products equal?

It is not possible to multiply more than two numbers at a time. If we have more than two numbers to multiply, we must group just two numbers.

In multiplying $2 \times 3 \times 4 = n$ we must multiply just two numbers at a time. To do this for $2 \times 3 \times 4 = n$ we can write

$$(2 \times 3) \times 4 = n\ .$$

The parentheses mean that we are grouping the 2 and 3 and we think of 2×3 as one number, 6 . Then the product is 6×4 or 24 . We could write:

$$2 \times (3 \times 4) = n\ .$$

This means we are grouping the 3 and 4. We think of this as one number 12. Then the product is 2×12 or 24.

$(2 \times 3) \times 4$ and $2 \times (3 \times 4)$ are each names for the same number, 24. The way in which we grouped the numbers did not change the product. When we group $2 \times 3 \times 4$ as $(2 \times 3) \times 4$ or as $2 \times (3 \times 4)$, we are using the <u>associative property of multiplication</u>.

2. If we use the associative property of multiplication to write $n = 3 \times 2 \times 5$, we write

$$n = (3 \times 2) \times 5$$
$$= 6 \times 5$$
$$= 30$$

or

$$n = 3 \times (2 \times 5)$$
$$= 3 \times 10$$
$$= 30 .$$

Find each product. Use the associative property of multiplication as was done above.

(a) $3 \times 2 \times 5$ $n = (3 \times 2) \times 5 = 6 \times 5 = 30$ or $n = 3 \times (2 \times 5) = 3 \times 10 = 30$

(b) $4 \times 2 \times 3$ $n = (4 \times 2) \times 3 = 8 \times 3 = 24$ or $n = 4 \times (2 \times 3) = 4 \times 6 = 24$

(c) $5 \times 2 \times 4$ $n = (5 \times 2) \times 4 = 10 \times 4 = 40$ or $n = 5 \times (2 \times 4) = 5 \times 8 = 40$

3. (a) Tell how to do this operation: $(2 \times 4) \times 5$
Multiply 2×4 first. Then multiply 8×5.

(b) Tell how to do this operation: $2 \times (4 \times 5)$
Multiply 4×5 first. Then multiply 2×20.

(c) Why must we group two of the numbers in multiplying?
We must group two numbers because multiplication is an operation on just two numbers.

4. (a) What is the result of $(3 \times 5) \times 2$? 30

(b) What is the result of $3 \times (5 \times 2)$? 30

(c) Is $(3 \times 5) \times 2 = 3 \times (5 \times 2)$? Yes

5. (a) Is $(3 \times 2) \times 4 = 3 \times (2 \times 4)$? Yes

(b) Are $(3 \times 2) \times 4$ and $3 \times (2 \times 4)$ different names for the same number? Yes

(c) In what way is $(3 \times 2) \times 4$ different from $3 \times (2 \times 4)$?
The factors are grouped differently.

Summary

We can multiply the three factors

2, 3, and 4 in that order in either of

two ways:

$(2 \times 3) \times 4$		$2 \times (3 \times 4)$
6×4	or	2×12
24		24

These two ways always give the same product.

When we replace one way by the other, we are

using the associative property of multiplication.

Because both groups of factors give the same

product, we can leave out parentheses and simply

write

$$2 \times 3 \times 4 = 24.$$

USING THE ASSOCIATIVE PROPERTY OF MULTIPLICATION

Exercise Set 20

Show how to use the associative property of multiplication to "prove" answers.

Example:	$3 \times 2 \times 5$	$= (3 \times 2) \times 5$
		$= 6 \times 5$
		$= 30$
	or	
	$3 \times 2 \times 5$	$= 3 \times (2 \times 5)$
		$= 3 \times 10$
		$= 30$

1. $2 \times 3 \times 4$
2. $4 \times 2 \times 6$
3. $3 \times 3 \times 2$
4. $4 \times 2 \times 3$
5. $2 \times 5 \times 4$
6. $3 \times 4 \times 2$

Use the associative property of multiplication to find these products.

Example:	3×40	$= 3 \times (4 \times 10)$
		$= (3 \times 4) \times 10$
		$= 12 \times 10$
		$= 120$

7. 5×50
8. 9×40
9. 6×80
10. 5×40
11. 6×50
12. 8×40
13. 3×70
14. 4×20
15. 5×90
16. 4×40
17. 7×80
18. 3×60

Answers for Exercise Set 20

1. $2 \times 3 \times 4 = (2 \times 3) \times 4 = 6 \times 4 = 24$ or $2 \times 3 \times 4 = 2 \times (3 \times 4) = 2 \times 12 = 24$

2. $4 \times 2 \times 6 = (4 \times 2) \times 6 = 8 \times 6 = 48$ or $4 \times 2 \times 6 = 4 \times (2 \times 6) = 4 \times 12 = 48$

3. $3 \times 3 \times 2 = (3 \times 3) \times 2 = 9 \times 2 = 18$ or $3 \times 3 \times 2 = 3 \times (3 \times 2) = 3 \times 6 = 18$

4. $4 \times 2 \times 3 = (4 \times 2) \times 3 = 8 \times 3 = 24$ or $4 \times 2 \times 3 = 4 \times (2 \times 3) = 4 \times 6 = 24$

5. $2 \times 5 \times 4 = (2 \times 5) \times 4 = 10 \times 4 = 40$ or $2 \times 5 \times 4 = 2 \times (5 \times 4) = 2 \times 20 = 40$

6. $3 \times 4 \times 2 = (3 \times 4) \times 2 = 12 \times 2 = 24$ or $3 \times 4 \times 2 = 3 \times (4 \times 2) = 3 \times 8 = 24$

7. $8 \times 30 = 8 \times (3 \times 10) = (8 \times 3) \times 10 = 24 \times 10 = 240$

8. $9 \times 40 = 9 \times (4 \times 10) = (9 \times 4) \times 10 = 36 \times 10 = 360$

9. $6 \times 80 = 6 \times (8 \times 10) = (6 \times 8) \times 10 = 48 \times 10 = 480$

10. $5 \times 40 = 5 \times (4 \times 10) = (5 \times 4) \times 10 = 20 \times 10 = 200$

11. $6 \times 60 = 6 \times (6 \times 10) = (6 \times 6) \times 10 = 36 \times 10 = 360$

12. $8 \times 40 = 8 \times (4 \times 10) = (8 \times 4) \times 10 = 32 \times 10 = 320$

13. $3 \times 70 = 3 \times (7 \times 10)$
$= (3 \times 7) \times 10$
$= 21 \times 10$
$= 210$

14. $4 \times 20 = 4 \times (2 \times 10)$
$= (4 \times 2) \times 10$
$= 8 \times 10$
$= 80$

15. $5 \times 90 = 5 \times (9 \times 10)$
$= (5 \times 9) \times 10$
$= 45 \times 10$
$= 450$

16. $4 \times 40 = 4 \times (4 \times 10)$
$= (4 \times 4) \times 10$
$= 16 \times 10$
$= 160$

17. $7 \times 80 = 7 \times (8 \times 10)$
$= (7 \times 8) \times 10$
$= 56 \times 10$
$= 560$

18. $3 \times 60 = 3 \times (6 \times 10)$
$= (3 \times 6) \times 10$
$= 18 \times 10$
$= 180$

USING THE PROPERTIES OF MULTIPLICATION

Exercise Set 21

1. Study the example below. Then copy the mathematical sentence filling in the unknown numbers. Write T if a statement is true and F if a statement is false.

Example: $(5 + 6) + 1 = 5 + (6 + 1)$

$11 + 1 = 5 + 7$

$12 = 12$ T

(a) $(2 \times 5) \times 3 = 2 \times (5 \times 3)$

$10 \times 3 = 2 \times$ __15__

$30 =$ __30__ ✓

(b) $(16 - 9) - 2 = 16 - (9 - 2)$

__7__ $- 2 = 16 -$ __7__

__[illegible]__ = __[illegible]__ F

(c) $(25 \div 5) \div 5 = 25 \div (5 \div 5)$

__5__ $\div 5 = 25 \div$ ____

____ = ____ F

2. Use the commutative and associative properties of multiplication to make the following products easier to find.

Example: $5 \times (2 \times 15) = 5 \times (2 \times 15) = (5 \times 2) \times 15$

$= 10 \times 15$

$= 150$

(a) $(2 \times 27) \times 5$ $(27 \times 2) \times 5 = 27 \times (2 \times 5) = 27 \times 10 = 270$

(b) $(11 \times 4) \times 5$ $11 \times (4 \times 5) = 11 \times 20 = 11 \times (10 \times 2) = (11 \times 10) \times 2 =$
$110 \times 2 = 220$

(c) $[(5 \times 5) \times 7] \times (2 \times 2)$ $[(5 \times 5) \times (2 \times 2)] \times 7 =$
$[(5 \times 2) \times (5 \times 2)] \times 7 =$
$[10 \times 10] \times 7 = 100 \times 7 = 700$

3. Use the commutative and associative properties of multiplication to find the following products.

(a) $20 \times 40 = (2 \times 10) \times (4 \times 10) = (2 \times 4) \times (10 \times 10) = 8 \times 100 = 800$

(b) $30 \times 30 = (3 \times 10) \times (3 \times 10) = (3 \times 3) \times (10 \times 10) = 9 \times 100 = 900$

(c) $30 \times 40 = (3 \times 10) \times (4 \times 10) = (3 \times 4) \times (10 \times 10) = 12 \times 100 = 1200$

4. Use the distributive property to show that $8 \times 25 = 200$.

$8 \times 25 = 8 \times (10 + 10 + 5) = (8 \times 10) + (8 \times 10) + (8 \times 5) = 80 + 80 + 40 = 200$

5. In a chart showing all products from 0×0 through 9×9, there are $10 \times 10 = 100$ multiplication facts. If you know the commutative property of multiplication, what part of the chart do you really need? The part on and above or on and below the diagonal from upper left to lower right. How many facts are given by this part of the chart? $(1 + 2 + 3 + \ldots + 10 = 55)$

DESCRIBING ARRANGEMENTS

Objective: To introduce arrangements with which we may associate a mathematical sentence of the form

$$a = (b \times c) + d$$

where a, b, c, and d refer to whole numbers

Vocabulary: arrange, arranged, arrangement

Teaching Procedure:

Overview: The material which concludes this chapter provides several "readiness" stages for that which we often refer to as "division with remainders." The intent of the present material is to emphasize ideas rather than skills. A more explicit study of "division with remainders" occurs in connection with Chapter 7.

The Exploration on pages 350-2 of the pupil text is to be used as the basis for initiating this "readiness" work. Before you begin this, be certain that children understand words such as arrange, arranged, and arrangement as applied to sets of objects, etc. As needed, illustrate with small sets arranged in various ways.

Precede the use of page 350 with any discussion of stamp collecting that may be appropriate for the children of your class.

The material on page 350 is entirely oral.

In connection with pages 350 and 352, you likely will find it helpful to draw on the chalkboard the arrangements for Sam's, Ellen and Janet's stamps.

Note that the arrangements of Joe's and Sam's stamps involve forming equivalent subsets in one way, and that the arrangements of Ellen's and Janet's stamps involve forming equivalent subsets in another way. Also note the deliberate absence of any "extra stamps" in connection with Janet's arrangements.

DESCRIBING ARRANGEMENTS

Exploration

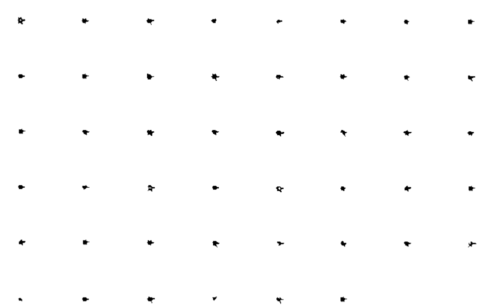
Joe, Sam, Ellen, and Janet were members of a stamp club. They each had some stamps to put in a stamp book.

Joe put 46 stamps on a page in his book.

He put 8 stamps in a row.

He made as many rows of 8 stamps as he could.

Here is a picture of the way Joe arranged his stamps.



1. (a) Are there 8 stamps on each row? No
   (b) How many rows of 8 stamps are there? 5
   (c) How many stamps are in the last row? 6

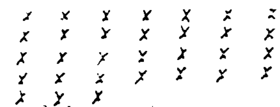
2. Tell if each of these sentences is true.
   (a) The 46 stamps are arranged in 5 rows of 8 stamps, with 6 stamps left over. True
   (b) The set of 46 stamps is arranged in 5 sets of 8 and a set of 6. True
   (c) $46 = (5 \times 8) + 6$. True

Sam put 31 stamps on a page in his book.

He put 7 stamps in a row.

He made as many rows of 7 stamps as he could.

3. Use counters or draw a picture to show how Sam arranged his 31 stamps.



4. Tell in several ways how Sam arranged his stamps.

There are 4 rows of 7 stamps, and 3 left over. There are 4 sets of 7, and a set of 3, etc.

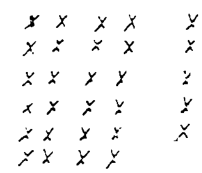
5. Complete this mathematical sentence to describe the way Sam arranged his stamps:

$$31 = (\overset{4}{\underline{?}} \times 7) + \overset{3}{\underline{?}}.$$

Ellen put 29 stamps on a page in her book.

She arranged as many of them as she could in 6 rows, with the same number of stamps in each row.

6. Use counters or draw a picture to show how Ellen arranged her stamps.



7. Complete each of these sentences to make it true.

(a) The 29 stamps were arranged as 6 rows of $\overset{4}{\underline{?}}$ stamps, with $\overset{5}{\underline{?}}$ stamps left over.

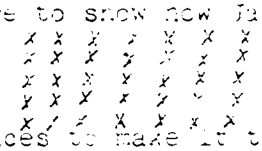
(b) The set of 29 stamps was arranged as 6 sets of $\overset{4}{\underline{?}}$ and a set of $\overset{5}{\underline{?}}$.

(c) $29 = (6 \times \overset{4}{\underline{?}}) + \overset{5}{\underline{?}}$.

Janet put 35 stamps on a page in her book.

She arranged as many of them as she could in 5 rows, with the same number of stamps in each row.

8. Use counters or draw a picture to show how Janet arranged her stamps.



9. Complete each of these sentences to make it true.

   (a) The 35 stamps were arranged as 5 rows of <u>7</u> ? stamps, with <u>0</u> ? stamps left over.

   (b) The set of 35 stamps could be arranged as 5 sets of [illegible] and a set having <u>0</u> ? members.

   (c) 35 = (5 [illegible]) + <u>0</u> ?

Use counters [illegible] drawings if needed to help you complete each of these [illegible] to make it true.

10. 25 = (<u>5</u> ? × [illegible]) + __.

11. 34 = (4 × __) + __.

12. 36 = (<u>4</u> ? × [illegible]) + <u>0</u> ?.

PARTITIONING SETS

Objectives: 1. To focus attention on two kinds of situations involving the partitioning of sets into equivalent subsets:

a. Those situations in which we know the number of members in the set and the number of members in each subset, and we wish to find the number of subsets that can be formed and the remainder (if any)

b. Those situations in which we know the number of members in the set and the number of subsets to be formed, and we wish to find the number of members that can be used in each subset and the remainder (if any)

2. To learn to associate with each kind of partitioning situation the mathematical sentence of the form

$$a = (b \times c) + d$$

where a, b, c, and d refer to whole numbers

Vocabulary: Partition, partitioning

Teaching Procedures:

Exploration:

Write this mathematical sentence on the chalkboard:

$$13 = (2 \times n) + r.$$

Have 13 books on your desk. Tell the children that you want to separate or partition the books into two piles, and that we can use the mathematical sentence on the chalkboard to describe how this is done.

Start by putting 1 book in each of two piles, with 11 remaining. Then write this beneath the first mathematical sentence:

$$13 = (2 \times 1) + 11.$$

Have one of the children explain how this sentence describes what has been done with the 13 books this far.

Continue partitioning the books into the two piles. Each time that 1 book is put with a pile, write on the chalkboard the mathematical sentence that describes the partitioning that far. Eventually the sequence of mathematical sentences will become:

$$13 = (2 \times 1) + 11$$
$$13 = (2 \times 2) + 9$$
$$13 = (2 \times 3) + 7$$
$$13 = (2 \times 4) + 5$$
$$13 = (2 \times 5) + 3$$
$$13 = (2 \times 6) + 1.$$

Emphasize that 6 is the most books we can put in each pile if we are to have the same number of books in each, and that there will be 1 extra book.

Help the children to see by using a mathematical sentence why there could not be as many as 7 books on each pile: If we wrote $13 = (2 \times 7) + r$, 2×7 already is more than 13.

Remind the children that in the work they have been doing, they have been partitioning a set into subsets.

It is expected that you will guide the children through the work on pages 355 and 356. Use diagrams of counters as needed in connection with the problems about Tom's and Betty's pencils.

Discuss orally the chart at the bottom of page 355 so that children understand how to interpret it.

Give particular attention to the interpretation of the two charts on page 356, with special emphasis on the instances where n is 0, 6, and 9. In your discussion of these charts be sure to cover these things:

a. For the chart at the left, what does each pair of whole numbers tell about partitioning Tom's pencils? Which pair of whole numbers should be used to answer the questions asked in the problem about Tom's pencils?

b. Do a similar thing with the chart at the right relating to Betty's pencils.

PARTITIONING SETS

Working Together

We can separate a set of things into subsets. When we do this, we may say that we <u>partition</u> the set.

We may try to partition a set so that each subset will have the same number of members.

Look carefully at the problems and the chart below. They may help us see two different ways to try to partition a set so that each subset will have the same number of members.

Tom had 23 pencils. He wanted to put them in 3 boxes, with the same number of pencils in each box. He wanted each box to have as many pencils as possible. How many pencils should he put in each box? How many extra pencils would he have?

Betty also had 23 pencils. She wanted to put them in small boxes, with 3 pencils in each box. She wanted to have as many of these boxes as she could. How many boxes of 3 pencils could she have? How many extra pencils would she have?

	Tom	Betty
Number of pencils in the set to be partitioned	23	23
Number of subsets, with the same number of pencils in each subset	3	n
Number of pencils in each subset	n	3
Number of pencils left over	r	r

We can use this mathematical sentence to help us think about partitioning Tom's set of pencils:

$23 = (3 \times n) + r.$

Copy and finish this chart to show pairs of whole numbers that will make the mathematical sentence true.

n	r
0	23
1	20
2	17
3	14
4	11
5	8
6	5
7	2
8	—
9	—

We can use this mathematical sentence to help us think about partitioning Betty's set of pencils:

$23 = (n \times 3) + r.$

Copy and finish this chart to show pairs of whole numbers that will make the mathematical sentence true.

n	r
0	23
1	20
2	17
3	14
4	11
5	8
6	5
7	2
8	—
9	—

Your teacher will talk with you about some of the things we have learned from these charts. After this, rewrite each mathematical sentence so that it will be true, using the greatest whole number you can for n.

$23 = (3 \times 7) + 2$

$23 = (7 \times 3) + 2$

MAKING A RECORD OF OUR THINKING

Objective: To introduce alternative algorisms (ways of recording steps in thinking) that may be used to help find replacements for n and r in mathematical sentences such as

$$32 = (5 \times n) + r$$

$$\text{or } 32 = (n \times 5) + r$$

To make each sentence true using the greatest possible whole number for n.

Vocabulary: Remainder

Teaching Procedure:

Overview: Children have been using drawings and counters to partition sets into equivalent subsets. They have seen that in mathematical sentences such as the above they have ways of describing these partitionings. In any situation there is a greatest whole number that can be used for n, if the sentence is to be true.

Children now need to have a more efficient way of dealing with such situations. Knowledge of the multiplication facts is helpful in deciding upon the greatest whole number to use for n. The remainder, r, then can be determined by subtraction.

Although some children can make such determinations "mentally," it is advantageous for all children to develop a written form for recording steps in their thinking. This is particularly true as numbers become larger and examples become more complex.

Two algorisms (forms of recording) are developed in connection with pages 359 and 360. We wish children to become acquainted with each, and then to select one as the preferred form. Some children may select one form; other children may select the other form. The selection should be made by each child individually in terms of his own preference. This choice should not be made immediately, however. At least a limited experience with each form is desirable, such as that provided on these two pages.

Exploration

Write this mathematical sentence on the chalkboard: $-2 = (b \times n) + r$

Raise the question of finding numbers to use for n and r so that (1) the sentence will be true, and (2) n will be the greatest possible whole number.

Develop your exploration of this along the same line as that used on page [illegible] of the pupil's book. Work first with one form of the algorism (Form I), then with the other (Form II). Introduce and use the word <u>remainder</u> as applied to the number 2 in your exploratory example: $-2 = (b \times 5) + 2$. Have children verify by drawings or with counters that the sentence is true.

Working Together

Guide the children through the work on pages 359 and 360. Develop each algorism at the chalkboard along with what is shown in the pupil material. Be sure to apply the term remainder to [illegible] each of the two partitionings developed on page 360.

Exercise Sets 21 and 22

Use these as soon as appropriate following the preceding Working Together section. In each set be certain that children understand the nature of the information given and needed to complete the chart. This can be clarified through the initial illustrative example.

It is expected that children will use one or another of the algorisms developed previously to help them complete the chart. Pupils may use one form for some examples and the other form for others. They should be encouraged to begin to move in the direction of the one they seem to prefer, however.

If some children are able to "figure things out in their heads" without using an algorism, that is fine. However, be sure these children understand and can use one or the other of the forms for recording their thinking. It will be more necessary for them to have this ability in connection with work later in the year.

In each Exercise Set you should give special attention to the interpretation of No. 10 when completed. In No. 10 of Set 22, no subsets of [illegible] can be made from a set of [illegible]. In No. 10 of Set 21, [illegible] objects cannot be partitioned into [illegible] subsets.

P201

MAKING A RECORD OF OUR THINKING

Working Together

We have been using mathematical sentences like these when we partition sets of things in certain ways:

$$47 = (n \times 6) + r$$

$$47 = (6 \times n) + r.$$

We try to make each sentence true using the greatest whole number we can for n. Knowing our multiplication facts will help us make true sentences in which we use the greatest whole number we can for n.

Look at this mathematical sentence: $47 = (n \times 6) + r.$

We know that $7 \times 6 = 42$ and that $8 \times 6 = 48$. So, 7 is the largest whole number we can use for n: $47 = (7 \times 6) + r.$

We now can think: $47 = 42 + r$, so we know that $r = 47 - 42$, or $r = 5$.

Now we can write: $47 = (7 \times 6) + 5$

We may call 5 the <u>remainder</u>.

Here are two ways to make a record of our thinking:

Form I

First write:	Then write:	Last write:
$6\overline{)47}$	$\begin{array}{r} 7 \\ 6\overline{)47} \\ \underline{42} \end{array}$	$\begin{array}{r} 7 \\ 6\overline{)47} \\ \underline{42} \\ 5 \end{array}$

Form II

First write:	Then write:	Last write:
$6\overline{)47}$	$\begin{array}{r} 6\overline{)47} \\ \underline{42} \end{array}$	$\begin{array}{rl} 6\overline{)47} & \\ \underline{42} & 7 \\ 5 & \end{array}$

Explain each of these ways to make a record of our thinking. *The explanation should bring out the kind of thinking illustrated in the middle of the page.*

Now let us see how we can use each of these ways to make a record of our thinking when we partition sets.

We have 27 oranges. We want to make as many bags of 4 oranges as we can.

$27 = (n \times 4) + r$

Let us use Form I to record our thinking.

$$\begin{array}{r} 6 \quad (n) \\ 4\overline{)27} \\ \underline{24} \\ 3 \quad (r) \end{array}$$

So, $27 = (6 \times 4) + 3$.

We now know that with the 27 oranges we can make __ bags of _4_ oranges each and we will have _3_ oranges left over.

Would we have found the same thing if we had used Form I instead of Form II to make a record of our thinking? *Yes*

We have 27 oranges. We want to put them in 4 bags, with the same number of oranges in each bag.

$27 = (4 \times n) + r$

Let us use Form II to record our thinking.

$$\begin{array}{r} 4\overline{)27} \\ \underline{24}\ 6 \\ 3 \end{array} \quad \begin{Bmatrix} (n) \\ (r) \end{Bmatrix}$$

So, $27 = (4 \times 6) + 3$.

We now know that with the _27_ oranges we can make _4_ bags of _6_ oranges each and we will have _3_ oranges left over.

Would we have found the same thing if we had used Form I instead of Form II in making a record of our thinking? *Yes*

As you work with more problems you likely will find that you like one method better than the other. Learn to use the one you prefer.

[illegible]

Copy and complete this chart.

Number in Whole Set	Number in Each Sub-set	Number of Subsets (n)	[illegible] (r)	Mathematical Sentence
Example:				
38				[illegible]
1. 34	[illegible]	5	2	34 = (5 × [illegible]) + 2
2. 42	[illegible]	6	0	42 = (6 × 7) + 0
3. 17	[illegible]	5	2	17 = (5 × 3) + 2
4. 43	[illegible]	[illegible]	7	43 = (4 × 9) + 7
5. 58	[illegible]	9	4	58 = (9 × 6) + 4
6. 14	[illegible]	[illegible]	0	14 = (7 × 2) + 0
7. 51	[illegible]	7	2	51 = (7 × 7) + 2
8. 29	[illegible]	[illegible]	5	29 = (3 × 8) + 5
9. 32	[illegible]	6	2	32 = (6 × 5) + 2
10. 3	[illegible]	0	3	3 = (0 × 4) + 3

Exercise Set [illegible]

Copy and complete this chart.

Number in Whole Set	Number of Subsets	Number in Each Subset (n)	Remainder (r)	Mathematical Sentence
Example: 48	9	5	3	48 = (9 × 5) + 3
1. 53	7	7	4	53 = (7 × 7) + 4
2. 39	8	4	7	39 = (8 × 4) + 7
3. 45	9	9	0	45 = (9 × 5) + 0
4. 27	4	6	3	27 = (4 × 6) + 3
5. 56	8	7	0	56 = (8 × 7) + 0
6. 43	5	8	3	43 = (5 × 8) + 3
7. 28	6	4	4	28 = (6 × 4) + 4
8. 36	7	5	1	36 = (7 × 5) + 1
9. 51	8	6	3	51 = (8 × 6) + 3
10. 3	5	0	3	3 = (5 × 0) + 3

Supplementary Exercise Set A

1. Complete the mathematical sentences below.

   a. [illegible] ÷ 9 = [illegible]   e. 5 = [illegible] ÷ [illegible]

   b. [illegible] = [illegible] 5   f. [illegible] 7 = [illegible]

   c. [illegible] − 9 = [illegible]   g. [illegible] = 56

   d. [illegible] = [illegible] ~   h. [illegible] = 60

2. Copy and complete the chart below.

Mathematical Sentence	Unknown Number
a. [illegible] p = 3[illegible]	a. p = 6
b. [illegible] n = 1[illegible]	b. n = 9
c. 1[illegible] ÷ p = [illegible]	c. p = 3
d. q ÷ [illegible] = [illegible]	d. q = 42
e. [illegible] = m	e. m = 81
f. [illegible] = 3[illegible]	f. w = 6
g. [illegible] y	g. y = 15
h. [illegible] ÷ [illegible] r	h. r = [illegible]

Supplementary Ex[illegible]se Set F

1. If n has [illegible] as a factor and [illegible] as a factor, what factors are [illegible] n [illegible] (... n, m, m ...)

2. a. How can you tell by looking [illegible] if it has [illegible] as a factor? (The last dig[illegible])

   b. [illegible] you replace [illegible] digit [illegible] by another digit and still have [illegible] with [illegible] as a fa[illegible] [illegible]

   c. Does [illegible] as a factor? (No, [illegible] digit is odd.)

3. You [illegible] a "star" multiplier. You can multiply any two numbers. You are asked to find the number of seats in a very large auditorium. The seats form an array. How would you do it? (Count the number of seats in a row, count the number of rows, then multiply these numbers.)

4. [illegible] marching band always forms an array when it marches. The leader likes to use many different formations. The band has [illegible] members in it. The leader is trying very hard to find one more member. WHY? (With 59 members, the band can march only in one column or in one row. With 60 members, there will be many possible formations.)

5. True or False? Every number with [illegible] as a factor has [illegible] as a factor also? (True) Explain your answer. ([illegible] 30 = [illegible] we can get [illegible] = 10 × 3.)

6. If neither [illegible] factor, then what do you know about the fa[illegible]? (Nothing at all. [illegible] has [illegible] as a factor but 6 does not.)

7. How would you use the fact, [illegible] × [illegible] to find the unknown factor in [illegible] × n [illegible] ([illegible] × 9 = (3 × 2 [illegible] = 3 × ([illegible] 9), [illegible] 3 × 3 [illegible])

You are told the multiplication fact $7 \times 15 = 105$.

a. Find 7×16 from 7×15. ($7 \times 16 = 7 \times (15 + 1)$
$= (7 \times 15) + (7 \times 1)$
$= 105 + 7$
$= 112$)

b. Can you find 7×18 without first finding 7×17? (Yes $7 \times 18 = 7 \times (16 + 2)$
$= (7 \times 16) + (7 \times 2)$
$= 112 + 14$
$= 126$)

c. Is there any correct way to fill the blank in

$7 \times$ ____ $= 101$. Why? (No, because $105 + 7 = 105$; 101 does not differ from 105 by 7.
[illegible] $7 \times 1 = 105$ [illegible])

4. Multiply a number by 4. Divide the product by 2. The result will have an even number. Why is this true?
(The operations amount to multiplying by 2.)

Supplementary Exercise [illegible]

1. Complete the following mathematical sentences.

a. [illegible] ÷ [illegible] = ~

b. [illegible] ÷ 8 =

c. ([illegible] 4) ÷ [illegible] =

d. ([illegible] ÷ 1[illegible]) ÷ [illegible] = 2

e. ([illegible] ÷ [illegible]) = 54

f. [illegible] ÷ [illegible] 6 =

2. Notice that [illegible] ÷ [illegible] ÷ [illegible] = [illegible] ÷ [illegible]

Also [illegible] ÷ [illegible] = [illegible] ÷ ([illegible]) = [illegible]

a. Is [illegible] ÷ [illegible] = ([illegible] ÷ [illegible]) [illegible] (Yes)

b. Is [illegible] ÷ [illegible] = ([illegible] ÷ [illegible]) ÷ [illegible] (Yes)

Can you see a general rule? (Dividing by 4 and then dividing the result by 3 amounts to dividing by 12, etc.)
Can you explain why it holds? (Perhaps the following will help explain. [(36 ÷ 4) ÷ 3] × 3 × 4 = (36 ÷ 4) × 4 = 36])

3. Use the trick we found in problem 2 to find

Example: 72 ÷ 24 [illegible] 72 ÷ 24 = [illegible] ÷ ([illegible] × 3)

= ([illegible] ÷ [illegible]) ÷ 3

= 9 ÷ 3

= 3.

a. [illegible] ÷ 1[illegible] = 80 ÷ (8 × 2) = (80 ÷ 8) ÷ 2 = 10 ÷ 2 = 5

b. 64 ÷ 16 = 64 ÷ (8 × 2) = (64 ÷ 8) ÷ 2 = 8 ÷ 2 = 4

c. 120 ÷ [illegible] = 120 ÷ (10 × 2) = (120 ÷ 10) ÷ 2 = 12 ÷ 2 = 6

d. 144 ÷ [illegible] = 144 ÷ (12 × 2) = (144 ÷ 12) ÷ 2 = 12 ÷ 2 = 6

[illegible]

Notice that (36 ÷ [illegible]) [illegible] = [illegible] = 1.

[illegible] [illegible] ÷ [illegible] ÷ [illegible] = [illegible] ÷ [illegible] = 1.

Here's another: (1[illegible] ÷ [illegible]) [illegible] = [illegible] = [illegible]

Also [illegible] ([illegible] ÷ [illegible]) = [illegible] = [illegible].

. In each exercise, give a simpl[illegible] way to get the same result
[illegible]. Multiply [illegible]. Then divide the result by [illegible]. (*Multiply*
[illegible]. Divide by [illegible]. Then multiply by [illegible]. (*Divide by 2.*)
. Divide by [illegible]. Then divide by [illegible]. (*Divide by 2.*)
. Multiply by [illegible]. Then multiply by [illegible]. (*Multiply by 6*
. Multiply by [illegible]. Then divide by [illegible]. (*Divide by 2.*)

. In each exercise, write a pair of numerals which correctl[y] fill the blanks. (The same number may be used twice, but do not use the number 1.)

a. ([illegible] × __3__) ÷ __2__ = [illegible] *also 27, 3; 36, 4, etc*

b. [illegible] ÷ __3__ × __27__ = [illegible] *also 2, 18; 6, 54.*

c. (1[illegible] ÷ __3__) ÷ __7__ = 1 *also 4, 3 and 2, 6 or 6.*

d. (3 × __2__) × __[illegible]__ = 36 *numbers with product*

. Use what you did in exercise [illegible] to find:

Example: [illegible] ÷ [illegible]

[illegible] ÷ 2[illegible] = ([illegible] + [illegible]) ÷ 2[illegible] = ([illegible] × 2) ÷ 2[illegible]

[illegible] ÷ 12

1[illegible]

[illegible]. [illegible] ÷ 1[illegible] = (72 × 2) ÷ 18 = 7[illegible] ÷ [illegible] = 8

[illegible]. 1[illegible] ÷ 1[illegible] = ([illegible] × 2) ÷ 16 = 72 ÷ 8 = 9

[illegible]. 1[illegible] ÷ 18 = (54 × 2) ÷ 18 = 5[illegible] ÷ 9 = 6

[illegible]. Write the numeral which correctly fills the blanks. In any one exercise, fill the blanks with the same numeral.

a. (32) ÷ [illegible] = [illegible]

b. ([illegible]) ÷ [illegible] ÷ [illegible] (The operations mean amount to dividing by [illegible] then multiplying by 2 [illegible] dividing by [illegible])

c. [illegible] ([illegible]) [illegible] ([illegible]) = [illegible] (Use associativity)

d. [illegible] (____) ÷ (____) [illegible] (No number can correctly fill the blank. Multiplying and then dividing by the same number will give [illegible])

e. [illegible] (6) [illegible] (2) = [illegible]

Hint: Use the distributive property.

f. ([illegible]) [illegible] (8) = [illegible] Commutative property also needed.

g. [illegible] (2) [illegible] (2) [illegible] = [illegible]

h. (5) [illegible] (5) [illegible]

i. ([illegible]) [illegible] + ([illegible] 7) + (3 × 7) = [illegible]

j. [illegible] ([illegible]) [illegible] ([illegible]) = [illegible]

k. ([illegible]) [illegible] ([illegible]) ÷ [illegible] = [illegible]

[illegible]

[illegible]plementary Exer[illegible]

1. The multipli[illegible] [illegible] [illegible] [illegible] [illegible] and [illegible] out [illegible] place. Put the correct numbe[illegible] in the left an[illegible] up[illegible]

		[illegible]	3	2
2		[illegible]		
0	0	0	0	0
[illegible]	1	[illegible]	3	[illegible]
[illegible]	[illegible]		[illegible]	[illegible]

2. n + [illegible] is odd and is less than [illegible]. The set of all factors of n has [illegible] members. What is n? [illegible], 35, 42 [illegible] are multiples of 7. Of these only 49 has 3 factors ([illegible], 7, 49). So n = 49.)

3. I'm thinking of a counting number. I will call it n.

   a. [illegible] n is an even number. (n must be even.)

   b. n × [illegible] < [illegible] (n is 0, 2, 4 or 6.)

   c. n has [illegible] as a factor.

      What is n? (Of these, only 6 has 3 as a factor. n = 6.)

4. Find a pattern [illegible] [illegible] [illegible] products

   [illegible] (Their successive differences are 4, 6, 8, 10, ...)

5. Find a pattern in the [illegible]

   1 × 3, [illegible] (Their successive differences are 5, [illegible], 9, 11, ...)

III. Use distributive property to find s in the following.

Example: [illegible] s

[illegible]

a) [illegible] × 18 [illegible] s — s = 126   k) [illegible] ÷ [illegible] s — s = [illegible]

b) [illegible] × 12 [illegible] s — s = 108   l) [illegible] ÷ [illegible] s — s = [illegible]

c) [illegible] × 18 [illegible] s — s = 126   m) [illegible] ÷ [illegible] s — s = 11

d) [illegible] × [illegible] s — s = 150   n) [illegible] ÷ [illegible] s — s = [illegible]

e) [illegible] × [illegible] s — s = 252   o) 72 ÷ [illegible] = s — s = [illegible]

f) [illegible] × [illegible] s — s = 144   p) [illegible] ÷ [illegible] s — s = 9

g) 12 × 12 = s — s = 144   q) 32 ÷ [illegible] = s — s = [illegible]

h) [illegible] × [illegible] s — s = 270   r) 24 ÷ [illegible] s — s = 6

i) [illegible] × [illegible] s — s = [illegible]   s) [illegible] ÷ [illegible] = s — s = 6

j) [illegible] × [illegible] s — s = [illegible]   t) [illegible] ÷ [illegible] = s — s = 6

IV. Find the unknown number in each of the following.

a) [illegible] − [illegible] = t — t = [illegible]   k) [illegible] + 15 + 29 = r — r = 81

b) [illegible] n [illegible] — n = [illegible]   l) 36 ÷ 9 = s — s = 4

c) [illegible] n — n = [illegible]   m) 309 + k = [illegible] — k = 157

d) [illegible] − [illegible] = a — a = 532   n) 9 × r = 72 — r = 8

e) [illegible] = t — t = [illegible]   o) 28 ÷ t = [illegible] — t = [illegible]

f) [illegible] ÷ [illegible] = [illegible] — [illegible]   p) a ÷ 6 = 7 — a = 42

g) 27 × [illegible] = t — t = 216   q) 118 × 5 = m — m = 590

h) [illegible] ÷ [illegible] = a — a = [illegible]   r) 54 ÷ p = 9 — p = 6

i) [illegible] = t — t = [illegible]   s) 28 + 14 + 73 = r — r = 115

j) [illegible] ÷ n [illegible] — n = 5   t) s ÷ 8 = [illegible] — s = 64

REVIEW

Set I

Part

1. Write the symbols for Union or Intersection needed to fill the blanks between the set names. Example: a) $\cap$

   Set A = {a, b, c, d, e, f}

   Set B = {e, f, g, h, i, j}

   Set C = {a, b, c, x, y, z}

   a) A $\underline{\ \cap\ }$ B = {e, f}

   b) B $\underline{\ \cap\ }$ C = { }

   c) A $\underline{\ \cup\ }$ C = {a, b, c, d, e, f, x, y, z}

   d) A $\underline{\ \cup\ }$ B = {a, b, c, d, e, f, g, h, i, j}

   e) A $\underline{\ \cap\ }$ C = {a, b, c}

   f) B $\underline{\ \cup\ }$ C = {a, b, c, e, f, g, h, i, j, x, y, z}

2. Write the word "even" or "odd" to make these sentences true. Example: a) even

   a) The sum of any two even numbers is an __even__ number.

   b) The sum of any two odd numbers is an __even__ number.

   c) The product of any even and odd number is an __even__ number.

   d) The sum of any odd and even number is an __odd__ number.

   e) The product of any two even numbers is an __even__ number

   f) The product of any two odd numbers is an __odd__ number.

3. Write the word "even" or the word "odd" to make these sentences true.

1) [illegible] n is an even number.
2) [illegible] n is an even number.
3) [illegible] n is an odd number.
4) [illegible] n is an even number.
5) [illegible] n is an [illegible] number.
6) [illegible] n is an even number.
7) [illegible] n is an [illegible] number.

4. Apply the "undoing" idea to these operations. Some are done for you.

Do	Undo
1) [illegible]	[illegible]
2) [illegible]	[illegible]
3) [illegible]	[illegible] ÷ [illegible]
4) [illegible]	[illegible]
5) [illegible] × 2	[illegible] × 2) ÷ 2
6) [illegible] ÷ [illegible]	(32 ÷ 4) × [illegible]
7) [illegible] × 2	[illegible]) ÷ 2
8) [illegible]	[illegible]
9) [illegible]	[illegible]
10) [illegible]	(2[illegible]) × [illegible]

5. Fill in the blanks.

1) [illegible] thousands + [illegible] hundreds + [illegible] tens + no ones = [illegible] thousands + ____ hundreds + [illegible] tens + no ones = [illegible] ________

c) 2 thousands + 24 hundreds + 6 tens + 9 ones =

__4__ thousands + 4 hundreds + 6 tens + 9 ones

The number is __4469__.

d) 4 hundreds + 18 tens + 11 ones =

__5__ hundreds + __9__ tens + 1 one

The number is __591__.

6. Read carefully.

a) Mary said, "When I work my arithmetic problems I use only the numerals 0, 1, 2, 3, 4 and 5." What number base does Mary use? (Base six)

b) Set B = {0, 1, 2, 3}
When Bill does his arithmetic homework he uses only the numerals in Set B. What number base does he use? (Base four)

c) I have the place values of three, nine, twenty-seven, eighty-one, ... etc. What number base is being used? (Base three)

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. On a spelling test of 50 words, Mary spelled 38 words correctly. How many words did Mary spell incorrectly?
(38 + n = 50. n = 12 Mary spelled 12 words incorrectly.)

2. There are six volleyballs in a carton. Our school buys four cartons of volleyballs. How many balls do they buy?
(6 × 4 = n n = 24 They buy 24 volleyballs.)

3. Ralph has a paper route. He delivers 82 papers each day. How many papers does he deliver on Monday through Friday? (5 x 82 = n n = 410 Ralph will deliver 410 papers.)

4. Dick wanted a baseball cap that cost $1.25, a bat that cost $2.15, and a glove that cost $4.68. How much money would the three items cost?
($1.25 + $2.15 + $4.68 = n n = $8.07 The three items would cost $8.07.)

5. Mr. Green drives 12 miles to work each morning. He drives 12 miles home each evening. How many miles does he drive each day? How many miles does he drive in a 5 day work week? (12 + 12 = d d = 24 5 x 24 = n n = 120 Mr. Green drives 120 miles each week.)

6. Janice is four feet three inches tall. Janice is how many inches tall? (4 x 12 = n n = 48, 48 + 3 = n n = 51 or (4 x 12) + 3 = n n = 51 Janice was 51 inches tall.)

REVIEW

Set II

Part A

1. Using the symbols $>$, $<$, or $=$ make these true sentences.

a) $7 \underline{\ >\ } 6$	f) $120 \underline{\ <\ } 189 - 9$
b) $0 \underline{\ <\ } 1$	g) $6 + 8 \underline{\ <\ } 15$
c) $5 \times 5 \underline{\ <\ } 125$	h) $2 + 6 \underline{\ =\ } 2 + 6$
d) $8 + 199 \underline{\ <\ } 200 + 10$	i) $8 - 1 \underline{\ <\ } 3 \times 3$
e) $264 \underline{\ <\ } 265 - 2$	j) $5 - 2 + 3 \underline{\ =\ } 6 + 2 - 2$

2. Write these in base ten numerals.
   a) six thousand four hundred = 6400
   b) four thousand one = 4001
   c) seven hundred seven = 707
   d) one thousand ten = 1010
   e) nine hundred thirteen = 913
   f) one hundred eight = 108

3. Write each of these sets in set notation.
   a) The set of even numbers greater than 40 but less than 55. {42, 44, 46, 48, 50, 52, 54}
   b) The set of odd numbers less than 28 but greater than 16. {27, 25, 23, 21, 19, 17}
   c) The set of counting numbers between 45 and 51. {46, 47, 48, 49, 50}
   d) The set of whole numbers less than 12. {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
   e) The set of the days of the week whose names begin with the letter "S" {Saturday, Sunday}
   f) The set of children in this room who are two years old. { }

4. In the chart tell which property is illustrated by the number sentence at the left. Write the first letter of each word that names the property instead of writing the words. For example, write C P M for Commutative Property of Multiplication.

Number Sentence	Property Illustrated
a) $4 \times 3 = 3 \times 4$	C P M
b) $6 + 18 = 18 + 6$	C P A
c) $5 \times (2 \times 1) = (5 \times 2) \times 1$	A P M
d) $8 \times 3 = 3 \times 8$	C P M
e) $(9 + 1) + 6 = 9 + (1 + 6)$	A P A
f) $(2 \times 6) \times 2 = 2 \times (6 \times 2)$	A P M
g) $7 \times 13 = (7 \times 10) + (7 \times 3)$	D P M
h) $35 \div 5 = (30 \div 5) \div (5 \div 5)$	D P D

5. Find what number is represented by y in each of these. Tell what operation is needed to find y. Use A for addition, M for multiplication, S for subtraction, and D for division. Example a) is done for you.

a) $6 + 9 = y$ $y = 15$ A

b) $y = 8 \times 5$ $y = 40$ M

c) $19 - y = 14$ $y = 5$ S

d) $y + 9 = 19$ $y = 10$ S

e) $y = 8 + 4$ $y = 12$ A

f) $4 \times 6 = y$ $y = 24$ M

g) $24 \div 8 = y$ $y = 3$ D

h) $7 \times y = 21$ $y = 3$ D

6. A = {Joe, Jack, Tom}

   B = {David, Donald}

   What operation could you use to find the number of members in $A \cup B$? (Addition)

   Name the members of Set $A \cup B$. (Joe, Jack, Tom, Donald, David)

7. W = {1, 2, 3, 4}

   R = {0, 2, 5}

   Could you use addition to find the <u>number</u> of members in $W \cup R$? (No)

   Name the members of the Set $W \cup R$. (0, 1, 2, 3, 4, 5)

8. $N \cup P$ = {a, b, c, d, e, f, g}

   P = {a, b, c}

   Could you use subtraction to find the <u>number</u> of members in set N? (No)

   Name the members of Set N. (Can't be done)

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. On Saturday 60 people watched the Little League ball game. Forty-four people bought hot dogs. How many people did not buy hot dogs? ($n + 44 = 60$, $60 - 44 = n$
   16 people did not buy hot dogs.)

2. Mr. Brown took a trip. On Monday he drove 360 miles, Tuesday 419 miles, Wednesday 284 miles. How many miles did he drive altogether? ($360 + 419 + 284 = n$ $n = 1,0$[illegible]
   Mr. Brown drove 1,063 miles.)

3. Carol bought 3 pairs of anklets. She paid 99 cents in all. How much did each pair of anklets cost? ($3 \times n = 99$ $n = 33$ Each pair of anklets cost 33 cents.)

4. Louis has 15 gum drops. He shares them equally with two of his friends. How many gum drops does each child receive? ($3 \times n = 15$ or $15 \div 3 = n$ $n = 5$ Each child will get 5 gum drops.)

5. Jerry's mother sent him to the store to buy a loaf of bread for 31 cents a loaf, a can of corn for 23 cents and two candy bars for 5 cents each. How much money should Jerry pay the clerk? ($31 + 23 + 10 = n$ $n = 64$ Jerry should pay the clerk 64 cents.)
Jerry gave the clerk a one dollar bill. How much change should Jerry receive from the clerk? ($64 + n = 100$ or $100 - 64 = n$ $n = 36$ Jerry should receive 36 cents change from the clerk.)

REVIEW

Set III

Part A

1. Rename the following as base ten numerals.

   a) 6 hundreds + 3 tens + 5 ones = <u>635</u>

   b) 5 hundreds + 13 tens + 5 ones = <u>635</u>

   c) 46 hundreds + 0 tens + 4 ones = <u>4,604</u>

   d) 4 thousands + 6 hundreds + 0 tens + 4 ones = <u>4,604</u>

   e) 4 thousands + 12 hundreds + 6 tens + 5 ones = <u>5,265</u>

   f) 16 hundreds + 0 tens + 11 ones = <u>1,611</u>

2. In the chart below, tell which property is illustrated by the number sentence at the left. Write the first letter of each word that names the property instead of writing the words. For example, write A P M for Associative Property of Multiplication.

Number Sentence	Illustrated Property
a) $(4+5)+1 = 4+(5+1)$	A P A
b) $6\times(1\times2) = (6\times1)\times2$	A P M
c) $15+10 = 10+15$	C P A
d) $4\times5 = 5\times4$	C P M
e) $(3+12)+5 = 3+(12+5)$	A P A
f) $12+42 = 42+12$	C P A
g) $(4\times3)\times2 = 4\times(3\times2)$	A P M
h) $28\div7 = (14\div7)+(14\div7)$	D P D
i) $12\times4 = (10\times4)+(2\times4)$	D P M

[illegible] Perform the operation indicated. Tell whether the answer is an odd number or an even number.

Example:

a)	[illegible] - 21	[illegible]	even	e)	13 × 117	1521	odd
b)	16 × [illegible]	[illegible]	even	f)	14 + 5	19	odd
c)	22 + 14	[illegible]	even	g)	10 + 3	13	odd
d)	18 × 32	[illegible]	even	h)	472 × 3	1416	even

[illegible] After each statement write always true, sometimes true, or never true.

a) An even number added to an odd number is an odd number. always true

b) A whole number multiplied by a whole number is a whole number. always true

c) A whole number divided by a whole number is a whole number. sometimes true

d) When zero is added to a whole number the result is not a whole number. never true

e) If the order of adding two whole numbers is changed the sum is changed. never true

f) A whole number subtracted from a whole number is a whole number. sometimes true

g) An even number multiplied by an even number is an even number. always true

h) An even number added to an even number is an odd number. never true

i) If the order of subtracting one whole number from another whole number is changed the unknown addend is changed. sometimes true

j) If the order of multiplying two whole numbers is changed the product is changed. never true

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. When Bill went to the school party he counted the children there. After 14 children left he counted again and found 35 were still there. How many had Bill counted the first time? (35 + 14 = n n = 49 Bill counted 49 people at first.)

2. The 891 children from Grant School came to watch a program that 153 fifth graders at Lee School gave. How many children were in the auditorium during the program? (891 + 153 = n n = 1,044 There were 1,044 children in the auditorium.)

3. There are 420 children in the school. Sixty-four children are in fourth grade. How many children are not in fourth grade? (64 + b = 420; 420 - 64 = b b = 356 356 children are not in fourth grade.)

4. Miss Reed has 33 children in her class. Seventeen children are girls. How many boys are in the class? (17 + b = 33; or 33 - 17 = b b = 16 There are 16 boys in the class.)

5. John has 116 marbles. His younger brother has 77 marbles.

   How many marbles do the boys have altogether? (116 + 77 = m m = 193 The boys have 193 marbles altogether.)

   How many more marbles does John have than his brother? (77 + n = 116; 116 - 77 = n n = 39 John has 39 more marbles.)

6. At a Little League candy sale, fifty boxes of candy were sold one day. Mark sold five boxes, Larry sold four boxes, and Tim sold eight boxes. How much candy did the three boys sell? (4 + 5 + 8 = s s = 17 The three boys sold 17 boxes of candy.)

   How many boxes of candy did the other boys sell? (17 + t = 50; 50 - 17 = t t = 33 The others sold 33 boxes of candy.)

SUGGESTED ACTIVITIES

Relay Problems

Problem Solution Progression

The first member of each team works one step of a given problem on the board, then returns to line. The next child works step two, etc. Any child detecting an error by a team mate may make corrections before his computation is made. The first team to complete a problem with the correct answer is the winner.

Example showing the step completed:

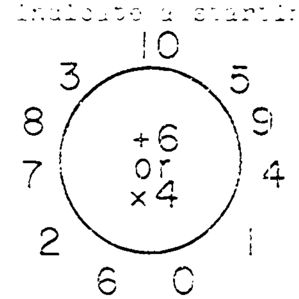
[illegible]

Range of difficulty may be adjusted to the ability of the class.

Game of Buzz as suggested in Chapter 4.

Clock Race

Draw a circle on the board. Write numerals from 0 through 10 around its face. Select a class member, indicate a starting point, and have him complete circling the face of the clock using the indicated process. Can be used for multiplication or addition.



[illegible]

1. Make up new symbols for a number system. Prepare some examples using your new symbols (a chart, number line, etc.) to show and explain to the class.

2. Look in the encyclopedia to find number systems used by other countries. Make a chart that explains one (or more) for display.

3. Make your own "magic" square, circle or triangle. Put it on the board for the class to prove.

Puzzles

1. Each of the squares below is not quite magic. One number must be changed in each. Cross off the number to be changed and write the correct number.

a.

12	5	10
11	13	3
4	~~12~~ 9	14

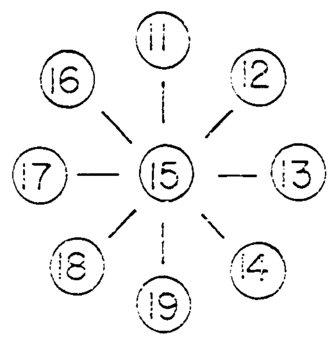
b.

10	~~21~~ 8	15
20	9	4
3	16	14

c.

8	7	15
17	10	~~2~~
5	13	12

2. In each circle, place one of the numbers 11, 12, 13, 14, 15, 16, 17, 18, 19. The sum of the three numbers in each line must be 45.



3. Find the Missing Digits . . . Don't give up too soon! For each of these examples write only one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 in each blank to make the example correct.

a) [illegible]

b) [illegible]

c) [illegible]

d) [illegible]

e) [illegible]

f) [illegible]

g) [illegible]

h) [illegible]

Brainteasers:

1. My brother is 5_ years old. On his next birthday he will be 10_{6}. What number base am I using? (Base Six)

2. John's dog weighed 7_{8} pounds on Monday. When he was weighed again two weeks later he weighed 11_{8} pounds! The dog had gained 2_{8} pounds. What number base is John using? (Base Eight)

3. The number of birds in a cage doubles every minute. The cage is full in half an hour. When was the cage half full? (It was $\frac{1}{2}$ full in 29 minutes.)

4. An ant is climbing a post thirty feet high. It climbs three feet every day and slips down two feet every night. How long does it take the ant to reach the top of the post? (28 days)

Chapter 5

SETS OF POINTS

PURPOSE OF UNIT

The purpose of this unit is to introduce some of the fundamental concepts of what might appropriately be called <u>physical geometry</u>.

The geometry which has traditionally been included in the curriculum of the elementary school and junior high school has been closely associated with the process of measurement. They have computed perimeters, areas, and volumes. In the high school, geometry is treated as a deductive system. Certain terms are taken as undefined, certain statements (axioms or postulates) about them are accepted without proof, and other terms are then defined and other statements (theorems) proved by the process of logical deduction.

Neither of these is the kind of development presented in this unit. The idea of measurement plays a very minor part, and is in fact used only in developing the concept of the circle. And while logical thinking is important in understanding mathematics at any level, the objective is not the development of geometric concepts and relations as a formal deductive system.

Rather, the object is to direct attention to what may be thought of as the geometric properties of familiar objects which do not depend upon measurement. We wish to begin the development of such mathematical ideas as point, space, line, curve, plane, angle, and simple closed (plane) curve. We shall study some of the representations of these in the things we see about us. We wish to develop the ability to use these concepts and their representations in understanding and in describing the world in which we live.

Certain familiar English words will appear in the unit which will be given mathematical meanings which are unfamiliar. Of these one of the most important is the word "point."

MATHEMATICAL BACKGROUND

If you look up the meaning of a word in a dictionary, you will find the word defined in terms of other words. If you continue looking up the words used in the definition, you soon might find one of these words defined in terms of the original word whose meaning you were seeking. This is known as a circular definition. In order for the dictionary to be helpful, you must know the meaning of some word in the circle prior to using the dictionary.

To avoid circular reasoning in mathematics, there are certain words which we do not attempt to define. Two of these words are point and line. We give meaning to these words by assuming certain properties concerning them. A statement which we assume to be true is called an axiom (or postulate). One axiom we might use is that through two points there is one and only one line.

The ideas of "circular reasoning" and "axioms" are not mentioned in the pupil text. Rather, a point is described as an exact or precise location. It cannot be seen or felt; it has no size. The teacher should remember that there are descriptions the pupil can understand but that we have not actually made a definition of a point. In fact, if you were to read the most advanced mathematics books in print you would not find a definition of a point.

Points are represented in an observable way by dots on paper and in a tangible way by a corner of a room where the two walls and the ceiling meet or as the end of a sharply pointed object. These are representations in the following sense. The dot made on a sheet of paper is merely an attempt to mark the localized geometric entity we call point. In fact the dot covers not one point but an infinite number (in this instance we mean more than [illegible]). When under some magnifying device such as a [illegible] the dot is clearly seen to cover many locations. [illegible]

Thus we assume that a point is a fixed location. A point does not move. If the dot made on a sheet of paper were erased, the location previously marked by this dot still would remain. Again if the sheet of paper were moved to some other pla[illegible] the point originally marked by the dot would remain fixed. Perhaps a more graphic demonstration of the permanency of the geometric point is given by the demolition of a building. The points occupied by each corner of each room remain unchanged. The difference is they are no longer represented by the physical objects called the corners. They would now have to be described by some set of directions leading to the location, such as 10 feet directly north of some marked point and then 10 feet up. Finally think of a pencil held in some position. Its tip represents a geometrical point. If the pencil is moved, its tip now represents a different geometrical point.

Once the geometrical meaning of point is understood, we are then prepared to define <u>geometrical</u> <u>space</u> or simply space as <u>the set of all points</u>. By this is meant the set of all conceivable locations in the universe. From what has been said about points, we realize that a physical object no matter how small already occupies more points in space than can be counted. Hence the set of all points must be a set all of whose members can never be counted. Geometrical space is neither empty space nor outer space. It is more inclusive. We must now think of exact locations. Physical objects as occupy only parts of space.

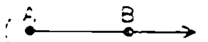
We next examine in both a cursory and intuitive fashion the idea of path between two points in space. This is a prelude to the important concept of line segment. A path is a particular set of points in space and may be thought of as the set passed through in going from one point to another. All such paths (whether "straight" or not) are thought of in geometry as <u>curves</u>. A stretched string between two points, a "curve" drawn on a sheet of paper, a route taken from one city to another are all

representations of curves. (The mathematical meaning of "curve" thus differs from the colloquial meaning.) From these examples we observe that a curve contains more points than can be counted.

A <u>line segment</u> (A———B, with symbol $\overline{AB}$) is that particular or special path or curve which may be represented by a string stretched tightly between two points. Another representation of a line segment would be a path drawn with a ruler and pencil connecting two points. The curve includes these points which are called the endpoints of the line segments. It can be thought of as the line of sight between two points and described as the most direct path. The line segment exists, of course, independently of any of its representations. For example, if the stretched string were removed, the line segment would remain since it is a set of locations.

A <u>line</u> ($\longleftarrow$A———B$\longrightarrow$, with symbol $\overleftrightarrow{AB}$) may be thought of as the extension in both directions of a line segment. It is a set of points which contains longer and longer line segments, but it can never be wholly represented by the drawing of line segments no matter how long. With an unlimited supply of rulers arranged so that lengths increase, we are able to represent longer and longer segments contained in a line. Beyond this our imagination must work to conceive of the unlimited nature of the line.

Nevertheless, from the representation of a line segment we are able to abstract certain properties of the line. The most important of these is that <u>precisely one line passes through two points</u>. In other words, given two points, there is exactly one line which contains both points. In the representation of points as dots it may be possible to draw two or more distinct lines between these points if the dots are not initially small enough. However, the realization that we are observing only an approximation of the idealized notions of point and line should clarify this apparent contradiction. We need only make smaller dots to force the conclusion we want.

The next section deals with the concept of ray. A <u>ray</u> (, with symbol $\overrightarrow{AB}$) is defined as a part of a line comprised of a point on the line called the <u>endpoint</u> of the ray and all points of the line in one direction from the endpoint. A beam of light emanating from a source of light is an excellent representation of a set of rays for which the source of light is the endpoint of each ray. From our knowledge of a line we can assert that a line contains more rays than can be counted, since any point of the line may serve as the endpoint of a ray on the line. However, since there are only two directions in a line from a fixed point on the line, there can be only two distinct rays on the line with the fixed point as common endpoint.

In a plane (hence in space) there is an unlimited number of rays with a common endpoint. Ultimately because of the uniqueness of the line through two points, we see that there is a unique ray (that is, one and only one) through two chosen points with one of the points as endpoint.

The <u>plane</u> is regarded as an unlimited extension of certain sets of points best represented by table tops, walls, floors, or any flat surface. The representations are in this instance only representations of parts of planes. An ever-growing table top provides a better and better representation but once again our imagination must come into play. Also we must remind ourselves that we are only representing certain sets of points in space. If the table is removed the set of points (locations) does not move.

A plane of which the table top is a partial representation can also be thought of as the set of all lines obtained by extending the line segments with endpoints on the table top. In this way it is clear that a plane contains more lines than can be counted. We observe, however, that if two points of a line are contained in a plane then the entire line is contained in the plane.

[illegible] observation of the representa- [illegible] entities--points, lines, and planes-- [illegible] relations:

1. [illegible] point in space there is an infinite [illegible] (by which we mean more than can be counted) [illegible], and through a line in space there is an infinite number of planes.

2. If [illegible] three points which are not on one line [illegible], then only one plane can pass through [illegible] sometimes express this fact by saying [illegible] points not on a line determine a unique plane." The example of a very thin book with its spine regarded as a segment, and its pages a multitude of planes containing the segment, is extremely helpful in establishing the relations mentioned.

The statement "three points not on a line determine a unique plane" is an example of an axiom. (Axioms are discussed on page [illegible].) The statement "two points determine a [illegible] line" is also an axiom.

[illegible] is next given to the intersections of lines [illegible] recall that the intersection of two sets is the [illegible] of the members which are in both sets. Thus, the intersection of two sets of points consists of the points which [illegible] in both sets. Observation of representations of lines and planes [illegible] these more significant relations:

1. If two different lines in space cross, their intersection is one point.

2. If a line and a plane cross, the intersection is either one point or the entire line.

3. If two planes cross, the intersection is a line.

The next concept is that of simple closed curve. The concept of simple closed curve includes certain curves in space (for example, twisted ones) as well as curves in a plane. We will be interested in just simple closed plane curves. This latter name is probably too long and so we will use the phrase simple closed curve for plane curves instead of the longer (and more explicit) simple closed plane curve.

We first return to the concept of curve, and restrict the discussion to curves in a plane. We think of a picture of a curve as a set of points which can be traced without lifting the pencil from the paper. With this idea of a curve, we see that line segments and rays and lines are curves. We next consider the closed curve, a curve whose "first" and "last" points are the same. It is possible to begin tracing at any point of a closed curve and come back to the starting point without lifting the pencil. By simple closed curve is meant a set of points in a plane represented by a drawn path which begins and ends at the same point and does not intersect itself. Here we can loosely speak of going around the plane simple closed curve because we will return to the starting point. In so doing we observe that a point is passed through or encountered just once (except for the starting point.) Note that this type of path closes on itself (returns to the point where it started) without intersecting itself. We see from a representation that there is a part we might call the inside, there is the curve, and there is a part we might call the outside. To pass from the inside to the outside of the simple closed curve by way of points in the plane containing the curve we must cross the curve.

After some rather brief experimentation with representations of this collection of curves we focus our attention on two special types, the polygon and the circle. A polygon is defined as a simple closed curve which is the union of line segments. Familiar examples of these are the triangle, a polygon which is a union of three line segments and the quadrilateral which is the union of four line segments. A close

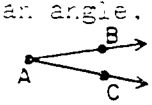
examination of the triangle reveals that the segments share endpoints in pairs and as a consequence there are three distinct endpoints which we call the vertices of the triangle.

After gaining some familiarity with the triangle and quadrilateral we turn our attention to another plane simple closed curve which cannot be accurately represented by ruler and pencil alone. This is the circle for which a compass is needed. From the representation of the circle made with the compass we observe that there is a point (where the metal point of the compass is placed) called the center, such that the distance from the center to each point on the circle is the same. This appears to be true because the distance between the metal point and the pencil point of the compass is not changed during the drawing. This leads to the characterization of the circle as a simple closed curve each of whose points is the same distance from a fixed point in the same plane called the center. Thus, the center is not a point on the circle. A line segment with one endpoint the center of a circle and the other endpoint on the circle is called a radius. Clearly all radii ("radii" is the plural of "radius") of a given circle are equal in length.

We have observed that a simple closed curve separates the plane into the part of the plane inside the curve, the curve itself, and the part of the plane outside the curve. We now call these parts of the plane the interior of the curve, the curve, and the exterior of the curve. The points on the curve itself are not in the interior and not in the exterior. We call the union of the set of points on the simple closed curve and the set of points in its interior a plane region. The two such regions determined by a triangle and a circle are called a triangular region and a circular region respectively.

The union of two rays which are not on the same line and which have a common endpoint is called an angle. The common endpoint is called the vertex of the angle. Although line segments

are not involved in the definition of an angle we see that two line segments with a common endpoint (but not on the same line) do in fact suggest an angle. For example, 

suggests the angle, 

Observe that the vertex of the angle is the common endpoint of the line segments. Also note that the segments are parts, but not all, of the suggested rays. This angle has a common endpoint as vertex and contains the segments. The segments are parts of rays in this case. Acceptance of this fact allows us to associate with a triangle three angles whose vertices are the three endpoints of the line segments which form the triangle. The three angles obviously are not part of the triangle but are merely associated with it. They explain the choice of the term "triangle" to name this figure. (A more apt choice for "triangle" might be "trilateral", but this latter term is not in use.)

TEACHING THE UNIT

LEARNING ABOUT SPACE

Objective: To introduce the study of geometry, and more specifically the unit, Sets of Points.

Materials: None

Vocabulary: geometry, imagination, Babylonians, pyramids, segments, polygons.

Suggested Teaching Procedures:

Each teacher may use his individual method of developing the concepts presented in this unit.

In preparing a lesson it is suggested that the teacher first read the information given in the Teacher Commentary and then carefully examine the pupil text.

In the pupil text certain sections are designated "Working Together." These are designed to be used in the class for development of an understanding or skill or to discover relationships. (You may, however, want the children to think through the exploration on their own at some other time.) The pupils are not to be expected to work through the exercises in "Working Together" independently. The teacher and pupils will do these exercises together, either with books opened or books closed as the teacher finds most effective. If pupil books are closed throughout this part, the teacher might say, "Now turn to page and look at 'Working Together.' Is this what we found to be true?" If the children have understood the ideas presented, then it is expected that little time will be spent on examining the "Working Together" section. That is, it is not intended that children tediously work through the exercises in Working Together if the ideas have previously been developed by the teacher when the pupils' books were closed. The pupils would then proceed directly to the Exercise Sets and would work independently.

The material in the "Working Together" sections is often written so that the pupil reaches a conclusion by guided discovery. The exercises for

the children to work independently al   labeled "Exercise Set   ." These exercises are intended to reinforce the "Working Together" sections. The answers should be discussed with the pupils after the children have completed work on these exercises.

For the first section, LEARNING ABOUT SPACE, the teacher might want to introduce the unit by using the material given on pages   and   of the pupil text before giving the pupils their texts.

Some children might want to find the word _geometry_ in a dictionary. A report on pyramids or on Euclid, the writer of the first geometry book," might be prepared by pupils. A review of sets language might be helpful. No answer, at this time is expected to the question, " n you guess what a set of 'points' would be?" This will be developed in the n t section, POINTS.

Chapter 5

SETS OF POINTS

LEARNING ABOUT SPACE

We are living in the space age. Man has already traveled in space and more space exploration will be done.

Suppose we were to plan a trip to Mars. Our space ship will have to follow a path which leads to Mars. Mars is moving all the time. To reach it, we must know its location in space, its speed in space, and its direction of travel.

The study of space and location is part of mathematics. This part of mathematics is called geometry. The things that we have been learning about the number system and about addition and subtraction belong to the part of mathematics called arithmetic.

To study geometry we need good imaginations. We make models and draw pictures to help us learn about things we cannot see. But our imaginations must help us too. Is your "imaginer" ready?

Geometry is not new. Thousands of years ago the Egyptians and Babylonians used ideas from geometry. It helped them to plan

pyramids, lay out their fields, and study about the moon, stars, and planets.

The first geometry book was written about 2200 years ago. It had most of the ideas we still use in studying about space. However, many new ideas about geometry are still being discovered. Maybe you will be one to discover a new idea.

At first "geometry" meant "earth measure." But now geometry also uses ideas which do not involve measurements. In this unit called Sets of Points we are going to study some of these ideas.

We know that a "set" is a collection of things. Can you guess what a set of "points" would be? First you would need to know what a "point" is. In this unit we will learn about points, space, curves, line segments, rays, circles, polygons, and angles.

POINT

[illegible]: [illegible] the understanding that a point may be [illegible] a position in space; also that a point has [illegible].

[illegible]: [illegible] pencil, [illegible] pencil, crayon, paper, chalk, [illegible] (directions given in page [illegible].)

[illegible]: point, exact location, exact position, represent.

[illegible] Teaching Procedures:

[illegible] introducing a new concept but we are [illegible] words such as point. We recognize the [illegible] that the word point means several things to [illegible]. However, we have in mind a particular [illegible], namely that of location. In fact we want to [illegible] of exact location. A point in this [illegible] will have no size at all. It is true we shall [illegible] represent points in an attempt to convey [illegible] exact location as nearly as possible by [illegible]. The smallest possible dot will be the best representation of a point. The descriptive [illegible] in the children's text is designed to clarify this new concept of point.

The children can work through Ex. 1-4 together with the pupil text open. Another way of proceeding would be for the teacher to develop the concept of point as an exact location before the pupils open their textbooks. This could be done by noting [illegible] points in the room, developing the ideas concerning point brought out in the Mathematical Background section.

Ex. [illegible] is quite important because it gets at the [illegible] that a point does not move.

It is important at this time to talk about the permanence of a location in space. To emphasize that points exist without physical objects occupying these locations, you can speak of a point 2 feet from a wall, 3 feet from an adjacent wall, and 5 feet from the floor. A more interesting question would be to ask if you wished to shoot down an enemy plane would [illegible] aim directly at the plane or [illegible] the point where the plane will [illegible] when the [illegible].

Do not spend too much time on the concepts of point and space. Some children will readily under[illegible] [illegible]. Others will

still require a point to have size and to be represented by physical objects. If, however, they will at least accept the idea that a point is smaller than anything heretofore discussed, they will probably have no difficulty with the subsequent material based on points. Their understanding of a point will continue to undergo refinement as progress in geometry continues.

Note that we use only capital letters to label a point.

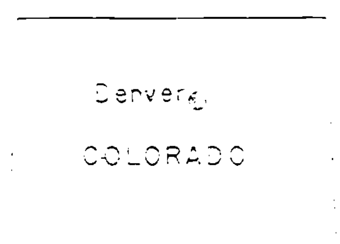
POINTS

What is a point? Is it the end of a sharp pencil? Is it the end of a needle? Is it the dot a pencil makes? Let's see what "point" means in geometry.

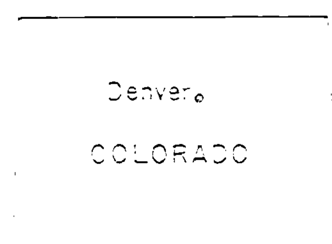
Working Together

1. Use your sharpest pencil to make a dot near the center of a sheet of paper. Now make a dot with a crayon. Next make a dot with a dull pencil. Do these dots look alike? (no) In what way are they different? In what way are they the same? They each mark a location. They are different in size and in location.

2. Which of these maps of Colorado best shows the location of the state capital? (b) Why? The dot is smaller and thus gives a more precise location.



(a)



(b)

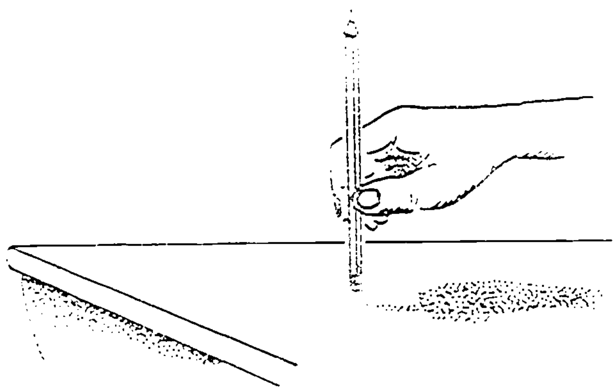
The dots you made in the first example and the dots in the maps of Colorado are attempts to show an exact location. The small dot marks the location more exactly. In geometry we often let a small dot represent a point. However, the dot is not the point any more than a picture of a cow is a cow.

A point in geometry means an exact location in space. [illegible] small [illegible] A point [illegible]

[illegible] However, [illegible] points [illegible] think of them. If you were going to mark [illegible] sheet of paper the location [illegible], you would [illegible] very sharp pencil. The dot made with the pencil would [illegible] position.

[illegible] represent or stand for a point [illegible] point is smaller than any dot which can be made.

[illegible] pencil with its tip [illegible] desk [illegible] in the [illegible]



[illegible] sharpened tip of the pencil show a point? [illegible] your pencil [illegible] this part of your desk. Does the tip now show a different point? Points do not move. They always stay in the same place.

In geometry we usually name pictures of points with capital letters like this:

A.　　　　　B.　　　　　C.

The points represented by A, B, and C can be called a "set of points." We will learn many interesting things about sets of points.

4. Describe a set of three points in your classroom.
(answers will vary.)

5. Describe a set of two points in your classroom.
(Answer will vary. Someone may even suggest a set with two of the points used in Ex. 4. If so, we recall something about sets within sets.)

6. Describe a set of eight points in your classroom.
(answers will vary. One example is suggested by the eight corners of some classrooms or of a box.)

Exercise Set 1

1. Which of these is the best representation of a point? (point B)

A• B· C○ D○

2. On your paper mark a set of five points using small dots. Label these pictures of points using the first five letters of the alphabet. (A. B. C. D. E. or any similar answer)

3. Write the letter of the best answer. (d)

   A dot made with a pencil covers

   a) one point

   b) one hundred points

   c) several points

   d) more points than can be counted

4. Which of these best describes a point? (c)

   a) a mark made with a pencil

   b) a very very small dot

   c) an exact location in space

   d) a dot

SPACE

Objective: To develop the idea that space is the set of all points, (i.e., the set of all locations.)

Materials: A glass or cup, several blocks of wood, a box of small grain, or other objects to clarify the idea of space, for example, a projection box.

Vocabulary: space, particle.

Suggested Teaching Procedure:

The teacher might start by a class discussion initiated as follows:

What is the smallest thing you know? You have heard of tiny particles of matter called atoms. No one has ever seen an atom even with a microscope. Yet an atom occupies space as we think of it. An atom is larger than a point because a point does not take up space. A point as we think of it has no size. An atom occupies more points of space than can be counted.

How many points are there in the space of our classroom?

a. More than one hundred?
b. More than one thousand?
c. More than you could ever count?

Answer: More than you could ever count.

What is space?

Allow time for children's ideas and discussions after space has been defined as the set of all points. Through these discussions the teacher will lead the following discoveries:

The inside of a basketball is an example of part of some space. Other examples are: the inside of a classroom, the atmosphere of the earth, or any location in the universe. The teacher should lead the child away from the child's idea that space is outside the earth's atmosphere; lead him toward the idea that <u>space</u> <u>is</u> <u>a</u> <u>set</u> <u>of</u> <u>all</u> <u>points</u>, that is all locations or positions. Any location in the universe is a point. Therefore, a glass or cup, an apple, a block of wood occupy points in space. Space is the set of all points (locations.)

After this preliminary discussion of space, the children's books might be opened and the "[illegible] Traveler" version examined. Such questions as, "What did they think that we now know about space?" would be appropriate.

SPACE

What is space? Is it air? Is it an empty place? Is it the distance from Earth to Mars? It is not any of these as we think of "space" in geometry.

Here are some examples of things which occupy sets of points in space:

The eraser on your pencil

The door to your classroom

Your little finger

Working Together

1. Now can you guess what "space" is? Which answer would you choose? (c)

(a) Space is something hollow.

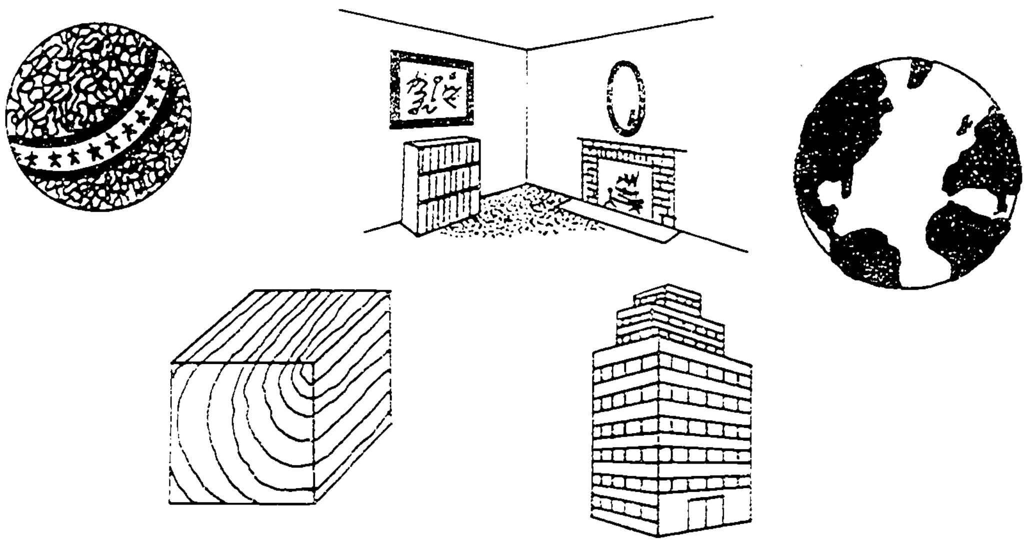
(b) Space is an object like a door or a finger.

(c) Space is the set of all the exact locations everywhere.

If you chose answer (c) you were correct. <u>Space is the set of all points</u>.

This means all exact locations everywhere. All the locations on the head of a pin, in your home, in your city and the sky above, in your country, in the world, and in the entire universe are points in space.

Space as we now picture it is probably very different from the idea you had. Any object you can think of covers or occupies lots of points of space. For example, a ball, a block of wood, a room, a building, the earth are all occupying parts of space.



2. Must a part of space be filled with air only? (no)

3. Does a block of wood contain one point of space, a thousand points of space, or more points of space than can be counted? (more points than can be counted.)

4. Place a cup on a desk. It represents many points. Move the cup to some other place. Does it now represent the same set of points as before? (no, because points do not move.)

5. Place a block of wood on a desk. It represents many points. Move it to some other place. Does it now represent the same set of points as before? (no, because points do not move.)

Exercise Set 1

Write the letter of the right answer.

1. Which of these best tells what space is? (c)

   a) Space is all empty places.

   b) Space is a set of points.

   c) Space is the set of all points.

   d) Space is the air around the earth.

2. In a truck load of grain, there are (b)

   a) just as many points as there are grains.

   b) more points than there are grains.

3. Which one of these represent a part of space? (a, c, d, f)

   a) A mark you make on your paper.

   b) The idea of truth.

   c) Your teacher.

   d) A tree.

   e) The idea of beauty.

   f) The crease in a piece of paper.

CURVES

Objectives: To develop the idea that a curve [illegible] suggested by the path traced by a pencil point [illegible] from one point to another [illegible] a line segment is a special kind of [illegible] the segment is [illegible] curve connecting the points.

Materials: String, [illegible] paper, pencils, [illegible] pinning board, [illegible] pipe [illegible] colored chalk, projection [illegible].

Vocabulary: curve, path, line segment, [illegible].

Suggested Teaching Procedure:

The development as contained in the pupil text can be followed. A peg board or a pinning board can be used for illustrating line segments by stretching a string tightly between two points. The globe or a map can be used by the children to show how one can get from one place to another-- thus tracing a path or curve.

In both Ex. 1 and 2 of "Working Together," think also of paths which are not entirely on the desk or the paper. That is, paths which leave the plane represented by the desk or the paper.

In Ex. 3, colored chalk might be used to supplement the development in the text to show the most direct path or curve (line segment) between two points marked on the chalkboard. Another way of representing a line segment is to fold a piece of paper. The crease made by the fold suggests a line segment.

In Ex. 4, it is suggested that pipe cleaners or wire be used to demonstrate the path.

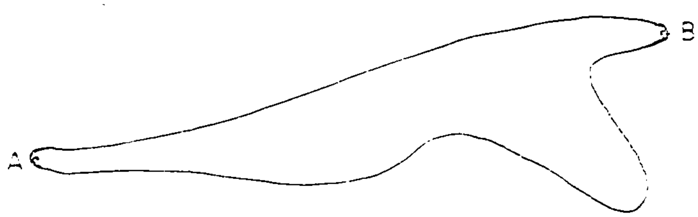
The symbol for line segment is one of several symbols used in this unit. The line segment in Ex. 4 could also be called line segment BA, written $\overline{BA}$. This method of naming line segments is used in Ex. 6.

CURVES

Working Together

1. Use two small bits of paper to mark two points on your desk. Trace with your finger to show ways you could go from one point to the other. How many different paths could you follow in going from one point to the other? (more ways than you can count) Can you trace the most direct way to go from one point to the other? yes, a line segment traces the most direct way from one point to another, but children do not yet know line segment)

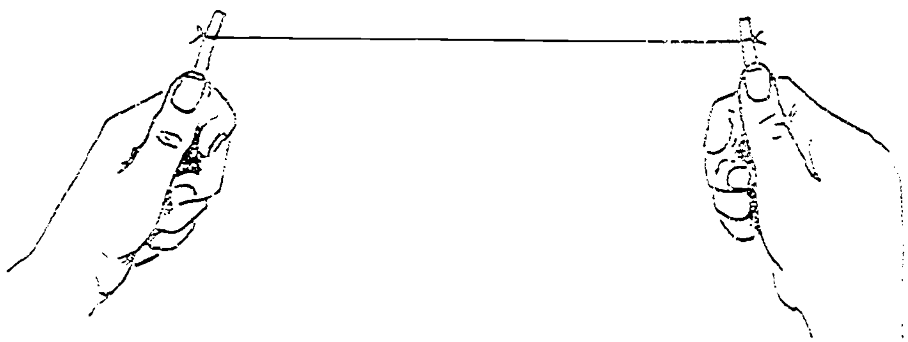
2. There are many ways of going from A to B. We see a picture of two ways.



Mark two points on a sheet of paper. Label them A and B. Show [illegible] ways of going from A to B on your paper. We do not have to stay on the paper. Think how you can go from A to B and touch the paper only at the dots. (this path would not be in the plane represented by the paper)

In going from one point to another, you have traced a curve with the tip of your finger or with your pencil. We think of a curve as a set of points. It is all the different locations your finger tip or pencil passes through in going from one point to another.

5. Let us think a little more about curves between two points. Suppose we use a piece of string tied to two pencils at the eraser ends to help us. We can let the erasers of the pencils mark two locations. Let us locate these points as far apart as the string will let us put our pencils. Does the string show the most direct path? yes



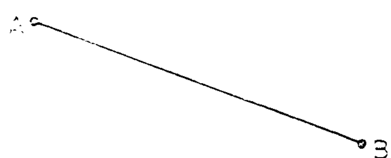
This direct path is a way of showing a special type of curve. We call it a line segment. Put dots on this string using chalk, pencil, or a pen. These dots mark points for us. We think of a line segment as a set of points. It is the set of all the points we have marked and all other points on our tightly stretched string. It also contains the two points represented by the erasers. We can show line segments in other ways.

[illegible]. Here is a block with the points A and B as shown in the picture.

Draw two curves on the block from point A to point B using two colors of crayon.

Did either curve you drew contain any line segments?
Answer will depend upon the curves the pupils draw.

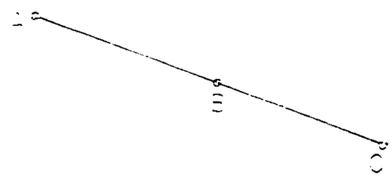
One excellent way to show a line segment is to draw a picture of one with a ruler and pencil. On your paper draw a segment connecting the two dots as shown in the figure below. We shall represent a line segment in this way.



We name this "line segment AB." A short way to write "the line segment AB" is $\overline{AB}$. $\overline{AB}$ means line segment AB. The line segment ends at points A and B. Therefore points A and B are called <u>endpoints</u>.

5. Think of the corner of your classroom as representing a point. What three things suggest line segments with this point as one endpoint? (the intersection of two walls or the intersection of a wall and the floor or the intersection of a wall and the ceiling.)

6. Name all the line segments you see represented in this figure with points in the set of points {A, B, C}.



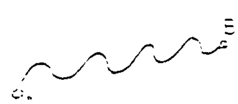
($\overline{AB}$, $\overline{AC}$, $\overline{BC}$, $\overline{AB}$ might also be named $\overline{BA}$, $\overline{AC}$ as $\overline{CA}$, and $\overline{BC}$ as $\overline{CB}$. There are 3 segments but six possible names for the three segments.)

7. Mark a point on your paper. Call it point A. How many different line segments can you draw with A as an endpoint?
more than can be counted

8. Give two examples of representation of line segments suggested by objects in your classroom.
answers will vary

9. Does your state have line segments as a part of its boundary?
This depends on the state.

10. Mark a point on your paper. Would you call your mark a line segment? (No, a line segment is not a point.)

11. Mark something like this  on your paper. Is it a line segment? (No)

12. Mark something on your paper which does not represent a line segment. (Answers will vary. Encourage them to do something different from the "wiggle" of ex. 11 or the dot of ex. 10.)

Exercise Set 5

1. Mark two points on your paper as is shown here.

   A $\bullet$ $\bullet$ B

   Draw three different curves from A to B.

   Put the letter of the best answer. d

   Each curve between these two points A and B goes through:

   a) one point

   b) three points

   c) many points

   d) more points than can be counted.

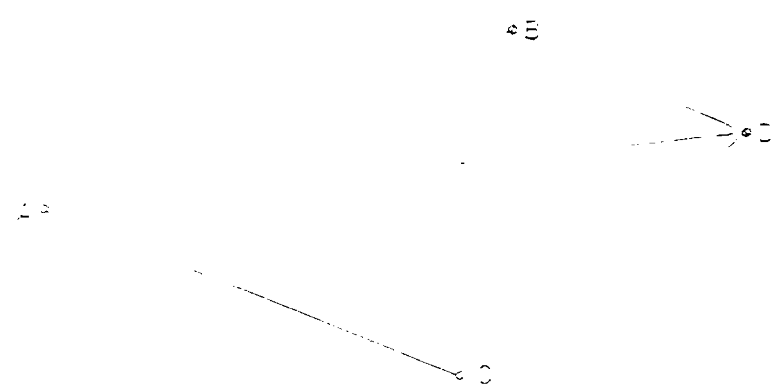
2. The set of points {A, B} is marked below. Copy this set on your paper. Draw as many line segments as possible having endpoints in this set. How many are there? one

   A $\bullet$

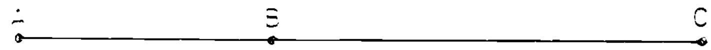
   $\bullet$ B

3. Copy the set of points {A, B, C} on your paper. Draw all the line segments having both endpoints in {A, B, C}. How many line segments are there? (three)

4. Copy the set of points {A, B, C, D} on your paper. How many line segments can you draw, each having two of the points named as endpoints? Be sure to draw all the line segments. (six) Name the line segments you drew. ($\overline{AB}$, $\overline{BC}$, $\overline{BD}$, $\overline{DC}$, $\overline{CA}$, $\overline{AD}$)



5. Name all the line segments you see in this figure. Both endpoints must be in the set, {A, B, C}. ($\overline{AB}$ or $\overline{BA}$, $\overline{AC}$ or $\overline{CA}$, $\overline{BC}$ or $\overline{CB}$ three line segments, each of which has 2 names)

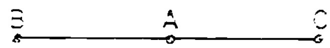


6. Mark a point on your paper and label it A as shown below. Draw pictures of two line segments having A as an endpoint.

one possibility

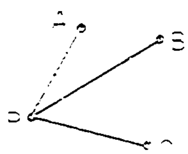
($\overline{AB}$ is one segment $\overline{AC}$ is another segment)

another possibility

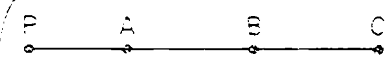


7. Mark a point on your paper and label it P. Draw pictures of three line segments having P as an endpoint.

one possibility

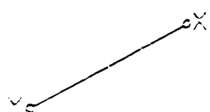


another possibility



(PA, PB and PC are line segments)

8. Complete these statements on your sheet of paper.



a. This is a picture of a line segment.

b. We write its name $\overline{YX}$ or $\overline{XY}$.

LINES

Objective: To develop the understanding that a line contains line segments of longer and longer length. A line has no endpoints. It extends infinitely far in two directions.

Materials: Paper, pencil, ruler, chalk.

Vocabulary: extend, describe.

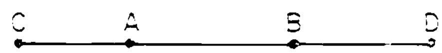
Suggested Teaching Procedures:

> The development as contained in the pupil text can be followed, the teacher and pupils working along together with the pupil text open.
>
> A different way of proceeding would be to have the pupil text closed as the ideas are developed by [illegible] work and pupil participation. The pupil book can then be examined, answering a question such as, "Is this a good record of what we did?"

Working Together

1. On your paper use a ruler to draw a line segment like $\overline{AB}$.

Draw a longer line segment which contains point A and point B by extending $\overline{AB}$ in both directions. Label the endpoints of this segment with the letters C and D. Does your drawing look something like this?



Is $\overline{AB}$ contained in $\overline{CD}$? yes

2. Draw an even longer line segment which contains points A and B by extending $\overline{CD}$ in both directions. Label the endpoints of this segment with the letters E and F. Your drawing might now look like this:

Is $\overline{AB}$ contained in $\overline{EF}$? yes

If you had a larger piece of paper and a longer ruler, you could draw a still longer line segment which would contain point A and point B. Think how this line segment would look if you were to draw longer and longer segments which contain points A and B.

Can you imagine how your drawing would look if it were extended without end? This is what we think of when we think of <u>line</u>. A <u>line</u> has no endpoints. It contains line segments of longer and longer length.

4. Below is a picture of a line.

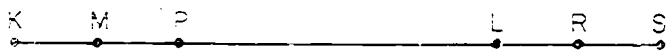
The arrows are used to show that it goes on and on in both directions without end. Only part of the line can be pictured on this page. We can call the line pictured line AB. A short way of writing line AB is $\overleftrightarrow{AB}$.

A and B name points on the line. We know C and D name other points on the line. We could also call this line, line CD, or line AC, or line AD.

Line AB is the same as line BA. What other names can this line have? Use just the points named.

$\overleftrightarrow{BC}$ and $\overleftrightarrow{CB}$, $\overleftrightarrow{BD}$ and $\overleftrightarrow{DB}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{DA}$, $\overleftrightarrow{DC}$

4. Here is a picture of $\overline{KS}$. M, P, L, and R, name other points of this line segment.



Copy $\overline{KS}$ and its labeled points.

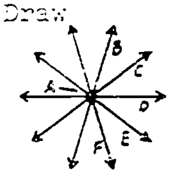
a) Draw a line segment which pictures still more of the line PS and also includes line segment KS. Can you draw a complete picture of the line PL? (no, the line extends without end)

b) Draw a picture of a line AB. On your paper show a short way of writing line AB. ($\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$)

Remember that a line segment is a set of many points and a line also is a set of many points.

5. Follow these instructions carefully.

a) Mark a point on your paper and label it A. Draw one line through point A. (Drawing will look something like this.



)

b) Now draw a different line through point A.

c) Next draw three more different lines through point A.

d) Mark one point different from point A on each of the lines you have drawn. Label the points with the letters B, C, D, E, F.

e) Can we draw more lines through point A? Yes, more than we can count.

f) Which is the correct ending to the sentence below.

Through point A we can draw: (more lines than you can count)

one line.

more lines than you can count.

g) Describe the position of a line segment through A which is not on your sheet of paper. (This segment would not be in the plane represented by the paper. There would be more of these line segments than you can count.)

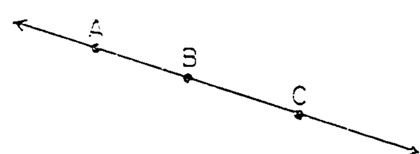
. On your paper mark two points, A and B.

a) How many line segments can you draw with endpoints A and B? (one)

b) How many line segments can you draw which pass through both A and B? (more than can be counted)

c) How many lines are there that contain both A and B? (one)

7. Below we have represented a line and three points of the line.



Shall we label this line $\overleftrightarrow{AB}$ or $\overleftrightarrow{AC}$?
(Either is correct. We could also label it $\overleftrightarrow{BA}$ or $\overleftrightarrow{CA}$ or $\overleftrightarrow{BC}$ or $\overleftrightarrow{CB}$)

In problem 6 we saw that with a ruler and a pencil only one line could be drawn through points A and B. From now on think of this statement as a fact; there is exactly one line that can be drawn through the two points A and B.

RAYS

Objective: To develop the understanding that a ray is part of a line. It is the union of its endpoint and all points on the line in one direction from this endpoint.

Materials: Pencil, paper, ruler, chalk, crayon.

Vocabulary: ray, symbol.

Suggested Teaching Procedures:

The development in the pupil text is in sufficient detail to follow. The teacher can develop the concepts using the chalkboard while the pupil books are closed. After this development, the pupil books can be opened and read to determine the answer to the question, "Does this tell what we did?"

The teacher may elect to proceed by working through with the pupils the "Working Together" section of the pupil text as a class activity with pupil texts open. The method of proceeding will be decided by the teacher as he knows which will be more effective with his group.

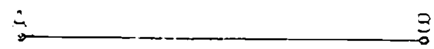
In Ex. 1 of the section "Working Together," the teacher might call attention that in the symbol for ray AB, the end without the arrow head is over the letter that names the endpoint of the ray.

Some children may need more practice similar to that in Ex. 2 to develop the concept of ray. The children might be interested in observing that a ray "begins" at the endpoint and "goes" in one direction without end. Of course, the ray does not actually move--we are talking about how we think about representing it on the chalkboard or a piece of paper.

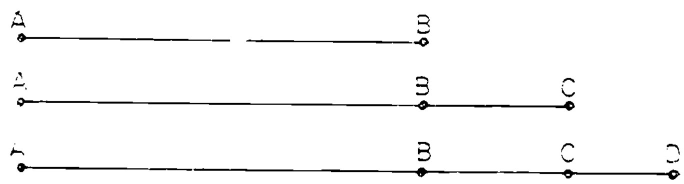
RAYS

Working Together

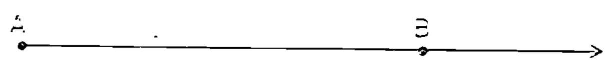
1. Use a ruler to draw a line segment AB on your paper.



Now suppose we make longer and longer line segments but always keep A as one of the endpoints, as



Then suppose we do not have a second endpoint, as in the picture below.



This gives us an idea for what is called a <u>ray</u>.

We can show a picture of only part of a ray on this page. We can name this ray, ray AB. Both A and B are points of the ray.

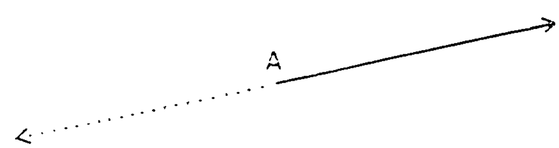
A ray has one endpoint. A is the name of the endpoint of the ray. A short way of writing ray AB is $\overrightarrow{AB}$. The endpoint is named first.

This is a picture of ray BA. What is its endpoint? (point B)

Ray BA is not the same as ray AB. Can you tell why? ($\overrightarrow{AB}$ and $\overrightarrow{BA}$ have different endpoints)
The endpoint of $\overrightarrow{BA}$ is B. What is the endpoint of $\overrightarrow{AB}$? (point A)

We can say that a ray is the union of the endpoint and all points on a line in one direction from this point.

For example, look at the line represented below and the point on it labeled A. One ray is represented by the solid part of the line. The other ray is represented by the dotted part of the line. The point A belongs to both rays represented and is called the endpoint of either ray.



A ray is always part of a line. A set of rays is nicely represented by a beam of light from a flashlight. Each starts at the flashlight and extends in one direction without end.

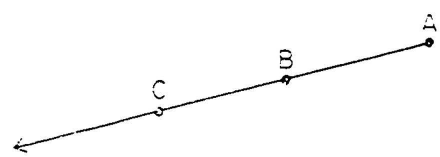
2. $\overrightarrow{AB}$ is represented below

   a) Is $\overrightarrow{AC}$ another name for this ray? (yes)

   b) Is $\overrightarrow{BC}$ another name for this ray? (no, since $\overrightarrow{BC}$ does not have A as its endpoint.)

   c) Is the ray $\overrightarrow{BA}$ represented? (no)

   d) Is the ray $\overrightarrow{BC}$ represented? (yes)

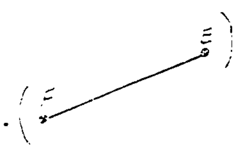


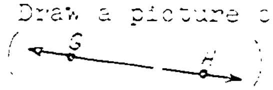
<u>Exercise Set</u>

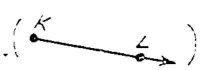
1. a) On your paper copy points F and E.

•E

•F

   Draw a picture of line segment FE. 

   b) Mark two points on your paper and name them G and H. Draw a picture of the line through G and H. 

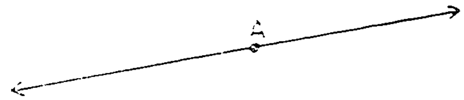
   c) Draw a picture of a ray. Name it $\overrightarrow{KL}$. 

   d) Write the symbol for line segment FE ($\overline{FE}$); for line GH ($\overleftrightarrow{GH}$); for ray KL. ($\overrightarrow{KL}$)

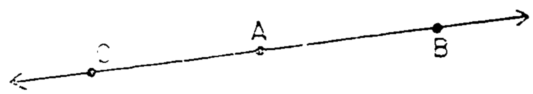
   e) What is the endpoint of $\overrightarrow{KL}$? (K)

   f) Is $\overrightarrow{KL}$ the same ray as $\overrightarrow{LK}$? (no) Why? (They do not have the same endpoint.)

4. Let A be the name of a point of the line below. How many rays which are part of this line can have A as an endpoint? (two)



5. Draw a picture of a line on your paper. Let A be a point of this line.



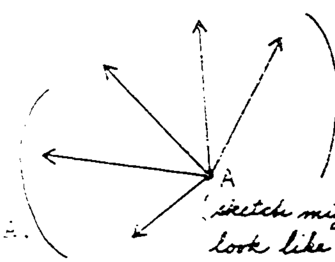
a) Choose a point of the line different from A in one direction and label it B.

b) Choose another point of the line in the other direction from A and label it C.

c) Name two rays with endpoint A which are part of this line. ($\overrightarrow{AB}$ and $\overrightarrow{AC}$)

d) Are there any more rays of this line which have A as an endpoint? (no)

6. Label a point on your paper as A.



(sketch might look like this)

a) Draw one ray with endpoint A.

b) Draw another ray with endpoint A.

c) Draw four more rays with endpoint A.

d) How many rays could there be with A as endpoint? (an infinite number - more than you can count)

[illegible] endpoint [illegible]

[illegible] paper draw [illegible]

[illegible]

[illegible] a thing [illegible] are

[illegible] more than can be counted.

[illegible]

[illegible] which contain point [illegible]
more than can be counted.

[illegible] draw which contain point [illegible]
more than can be counted.

[illegible] draw which contain [illegible]

[illegible] one

Which is the correct ending?

[illegible] A ray has [illegible]

a) exactly 1 endpoint. a) exactly 1 endpoint.

b) 2 endpoints. b) 2 endpoints.

c) no endpoints. c) no endpoints.

[illegible] A line segment has [illegible]

a) exactly 1 endpoint.

b) 2 endpoints.

c) no endpoints.

[illegible] that a part of a plane is [illegible] points [illegible] which are on a flat surface [illegible] paper lying flat, a table top, a [illegible], a wall; to develop the concept of plane from [illegible] part of a plane; to develop the idea that a plane contains [illegible] more lines than can [illegible].

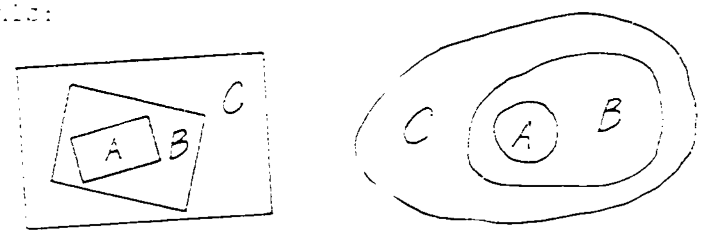
[illegible] pencils, crayons, [illegible] for each pupil.

[illegible] part of a plane, (region).

[illegible] Procedure:

The [illegible] may want to develop the idea of part of a plane [illegible] then whole plane as given in the pupil [illegible] pupils open their books. After this development, books may be opened to answer a question [illegible] "Is this a good record of what we have done?"

Ex. [illegible] "Working Together" is intended to develop the concept of plane. This is done in a way similar to the development of line. With line we started with a line segment and showed that longer and longer line segments could be drawn which contained a given two points. Then we imagined a line "segment" so long that it had no endpoints--and called this a line. With plane we start with a part of a plane and then draw larger and larger parts of a plane. Then we imagine a part of a plane (a table top) growing larger and larger without ever ending and we call this a plane. The children's drawings for Ex. [illegible] might look something like this:



Any drawing which shows region C larger than region B and region B larger than region [illegible] is satisfactory.

The word region is used here in an ordinary way. A mathematical meaning for region is developed in the [illegible], REGIONS IN A PLANE.

PLANES

Working Together

1. Can you find some flat surfaces in your classroom? Name as many as you can.
(desk top, floor, chalk board, window pane etc.)

Do you know the geometric name for a set of points suggested by a flat surface? It is plane. Each flat surface you have named represents part of a plane.

2. Put your finger on a point on the top of your desk. Now put it on a different point. How many different points can you find on the flat top of your desk? (more than can be counted) How many points do you think there are in a plane? (more than can be counted)

Put your finger at a point above the top of your desk. Then at a point below the top of your desk. Are there many points which are not in the plane represented by the top of your desk?

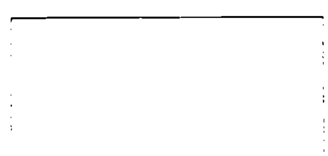
Yes, more than we can count.

From now on we shall think of a part of a plane as a set of points in space. It is the kind of set suggested by all points on a flat table top, or on a wall, or on the floor. A piece of paper lying flat on your desk also suggests a part of a plane.

To get a better idea of what we mean by a plane, follow these directions.

a) Use the picture of the figure below. Draw one like it near the center of a piece of paper.

See teacher commentary page



a) Trace the figure with a (red) crayon. Color the part of the plane inside of the figure the same color. Give this colored part the name A. Is this colored region a picture of part of a plane? yes.

b) Draw a bigger figure which encloses the colored region.

A. Color the larger figure and its inside (red) also. Name this new colored region, B.

Does the new colored region picture a part of the same plane as A? yes.

Does colored region A or colored region B picture more of this plane? region B.

c) Draw a third figure which encloses the colored region B. Color this figure and its inside (red). Name this new colored region, C.

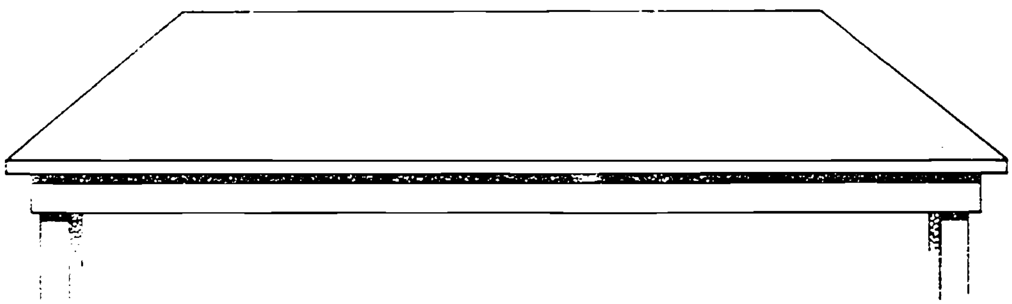
Does this new colored region picture a part of the same plane that A did? yes.

Does colored region A or colored region B or colored region C picture more of this plane? region C.

d) Can you draw a picture of the complete plane suggested by regions A, B, and C?

No. A plane extends without end.

As we think of a line containing longer and longer segments so shall we think of a whole plane as containing larger and larger flat surfaces. Imagine your table top growing larger and larger on all sides. You would then have a table top upon which you could walk as far as you wishes in any direction.



5. Does the set of points represented by the table top move when the table is moved?
(No. Points or sets of points do not move.)

6. Name some other objects which represent parts of planes. (Answers will be many and varied.)

7. Is there more than one plane in space?
(Yes. There are more than can be counted.)

We shall often use a sheet of paper placed on a flat table or desk top to represent a part of a plane. The table top itself may be thought of as containing even more points of this same plane.

Answer Key [illegible]

1. [illegible] think of it [illegible] points than can be counted?

more than can be counted

2. [illegible] sheet of paper. Think of it as part of [illegible] Is it possible to draw more than one line on [illegible] plane? yes

[illegible], draw two lines. Pupils lines will be varied. Some may intersect. Pupils' papers should show fifteen lines.

[illegible] draw three more lines.

Draw ten more lines.

3. Does a plane contain one line, [illegible] lines, or more lines than can be counted?

more than can be counted

4. Think of the top of your [illegible] as part of a plane.

Describe the location of a point not on this plane.

Describe the location of a line not on this plane.

Accept all reasonable answers. For example, if the representation of the plane is [illegible], point above or below it [illegible]. The same is true for lines.

5. Does a plane contain all points of space? (no)

6. Does a plane contain some lines but not all lines in space? yes

7. Take a sheet of paper. Think of it as part of a plane. Describe a line which is not on this plane. Draw a ray which is on the plane. (⟶ Answers will vary.) Describe a ray which is not on this plane. (Answers will vary.)

8. If the endpoint of a ray is not on a certain plane, is the ray on that plane? (No)

9. If the endpoint of a ray is on a certain plane, then must the ray be on that plane? (no)

10. If two points of a ray are on the plane, then must the ray be on the plane? (yes)

LINES AND PLANES

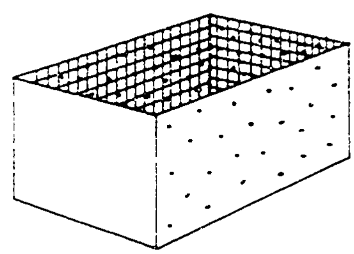
Objective: To develop the understanding that:

1. If two points of a line are contained in a certain plane, the whole line is contained in that plane.
2. Through two points there can pass more planes than can be counted.
3. Through three points not in a line there can be one and only one plane.

Materials: Cardboard and paper which can be used to represent planes; knitting needles or sticks which can be used to represent lines; a projection box with bamboo sticks or drinking straws or some other representation of lines.

A projection box that would be helpful in developing understanding can easily be made.

Take any sturdy cardboard box and punch holes in the top and three sides as shown in the sketch. By using cards to represent planes and bamboo sticks, straws, or wires to represent lines, we can, by placing these through the holes indicate intersection of lines, show that three points not in a line determine a plane, and illustrate that many planes can pass through one line.



Vocabulary: [illegible], observation, geometric.

Suggested Teaching Procedures:

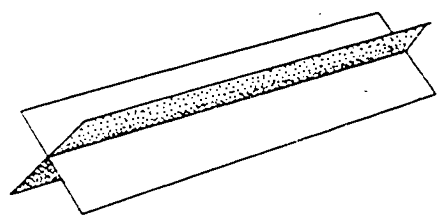
Ex. 1 and 2 of "Working Together" develop the idea that if two points of a line are in a plane, then the whole line is in that same plane. Many

examples similar to Ex. 1 will be needed. This section and the next on INTERSECTIONS OF LINES AND PLANES will need many concrete demonstrations and activities in order for children to grasp the rather difficult concepts presented.

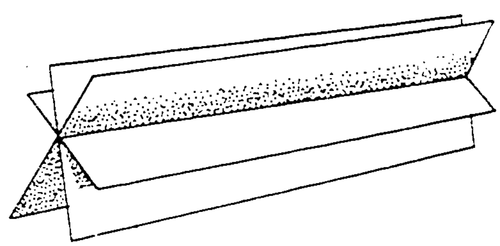
To further the concept discussed in Ex. 1, choose two points on a wall, on a book cover, on a side of a desk, in the projection box, on a face of the projection box, on a piece of cardboard, etc.

Ex. 3 - 1[illegible] develop the understanding that an infinite number of planes can pass through two points (and hence an infinite number of planes can pass through one line.)

Again many examples will need to be given. If each child has several pieces of cardboard to represent planes, he can arrange them to show a number of planes passing through one line. For example, in Ex. [illegible] he might hold them as shown.



Another way to show this idea would be to cut a slit in a cardboard. The slit represents a line. Through this slit pass a number of cardboards in different positions as shown.



Children need many opportunities to handle cards representing planes in order to become familiar with the notion "Through a line there are more planes than can be counted." [illegible] pupils will merely "parrot" this concept and not really understand the idea involved.

Ex. 11 Develops the idea that three points, not on a line, determine one and only one plane. Pupils can participate in many activities similar to a) and b), to gain the understanding that one and only one plane can pass through three points not on the same line. A flip-top chalkboard is an excellent device to illustrate this concept.

LINES AND PLANES

Working Together

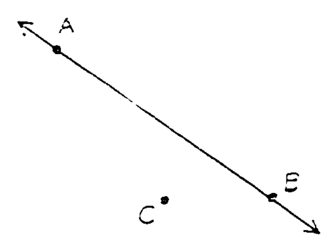
Let us think about a line and a plane. Suppose the line has two of its points in the plane. For example, think of the points A and B represented on this page. The page suggests part of a plane which contains A and B.

A •   • B

1. Answer these questions carefully.

a) How many lines contain both points A and B? (one)

b) Are all of the points of $\overleftrightarrow{AB}$ contained in a plane which contains A and B? (yes)



c) Think of a third point in the plane and label it C. Draw line CA. Is $\overleftrightarrow{CA}$ in the plane? (yes)

d) Draw line CB. Is $\overleftrightarrow{CB}$ in the plane? (yes)

Suppose we have two points, A and B. Suppose we have a plane called E. If point A is in plane E and point B is in plane E, then the entire line AB is in plane E.

2. Which is the correct ending: (b)

A line with two of its points contained in a plane

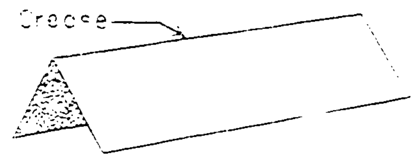
a) has some, but not all, of its points contained in that plane.

b) has all of its points contained in that plane.

1. Is there more than one plane containing points A, B? If there is more than one plane, give an example. Remember that $\overline{AB}$ is just the line containing the two points.

there are more than can be counted

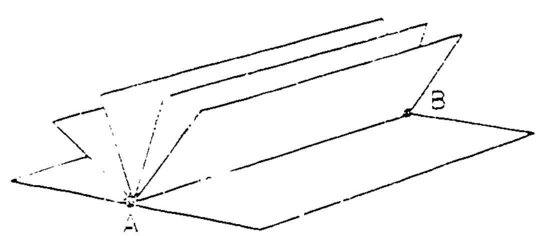
2. Fold a piece of paper in half. We think of the crease as a line segment. Stand the folded paper on your desk so that the crease does not touch it. The paper makes a tent.



Does this suggest parts of two planes which contain the line segment represented by the crease? yes If so, show them.

3. Give an example of two points and three planes passing through them. accept any reasonable answer

4. Open a thin book so that you see the pages as in the figure below. The spine of the book suggests a segment. Name it $\overline{AB}$.



a) What does each page suggest? (a part of a plane)

b) Does each page pass through the spine of the book? (yes)

7. Choose two points in space. Now how many planes do you think contain the same two points? (more than can be counted)

8. Choose a line segment in space. How many planes do you think contain this line segment? (more than can be counted)

9. Choose a line in space. How many planes do you think contain this line? (more than can be counted)

10. Which is the correct ending?

a) Two points in space are contained in (3)

1) only one plane.

2) many, many planes, but we could count them.

3) more planes than can be counted.

b) A line segment is contained in (3)

1) only one plane.

2) many, many planes, but we could count them.

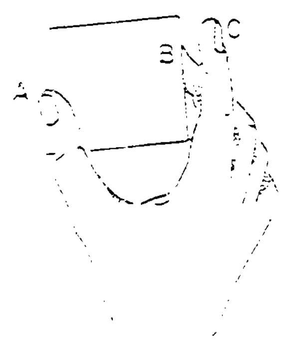
3) more planes than can be counted.

c) A line is contained in (3)

1) only one plane

2) many, many planes, but we could count them.

3) more planes than can be counted.


A
B
C

[illegible]) Think of [illegible] representing part of a plane and the hinges representing two points. As the door swings open, [illegible] it suggest many planes through these points? (yes)

[illegible] finger tip against the door. Your finger tip represents [illegible] third point which holds the door open in [illegible] position.

This again suggests that through three points not on a line there passes just one plane.

[illegible] Review

[illegible]) A plane [illegible] more points and more lines than can [illegible].

[illegible]) If two points of a line are contained in a plane, the whole line is contained in the plane.

[illegible]) Through two points there passes more planes than can [illegible].

[illegible]) Through three points not on a line there passes one [illegible] plane.

INTERSECTIONS OF LINES AND PLANES

Objective: To develop the understanding that:

1. If two different lines in a plane cross, their intersection is one point.

2. If two different planes in space cross, their intersection is one line.

3. If a line and a plane cross, the intersection is either one point or the entire line.

Materials: Cardboard or paper which can be used to represent planes; pencils, wires, sticks, or straws which can be used to represent lines; a projection box.

Vocabulary: Intersection, union, ceiling.

Suggested Teaching Procedures:

The concept of intersection of lines and planes can be a difficult one for children--and adults! The wide use of concrete materials will prove most helpful.

Ex. 1 - 3 in "Working Together" review intersection and union. Ex. [illegible] - [illegible] show that the intersection of two lines is a point. Ex. [illegible] shows that the intersection of two lines can be the empty set if the lines are parallel or "skewed."

The projection box can show these ideas rather clearly.

Ex. [illegible] show that the intersection of a line and a plane

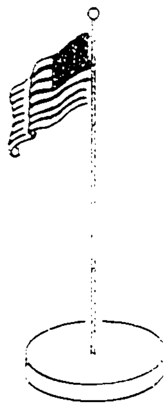
a) can be one point (the pencil need not be perpendicular to the paper, of course.)

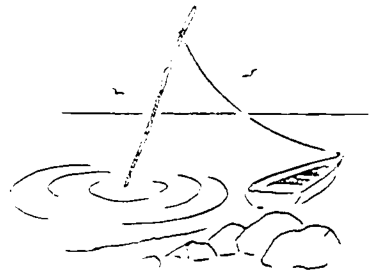
b) can be no points (the pencil and the cardboard are in parallel planes.)

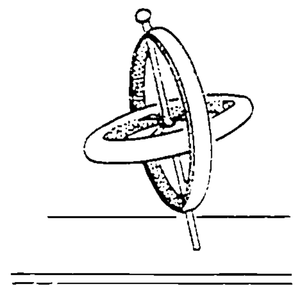
c) could never be just two points.

d) can be many points (the line would lie in the plane.)

Sketches such as these might help show the intersection of a line and plane:







... 11 [illegible] the intersection of two planes [illegible]:

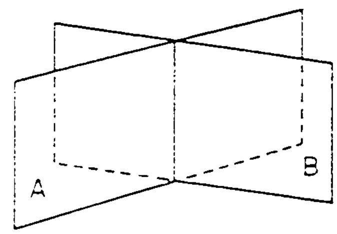
(a) Has just one point (There is no way for two planes to intersect so that their intersection is just one point.)

(b) Has just two points (ditto)

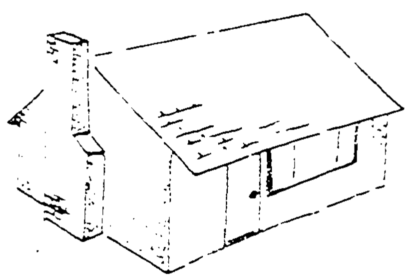
(c) More than two points (The intersection of two planes will always be in a line, unless the planes are parallel, in which case their intersection would be the empty set.)

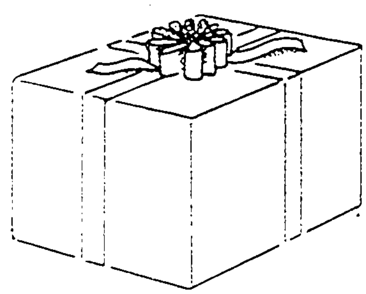
(d) [illegible] lines.

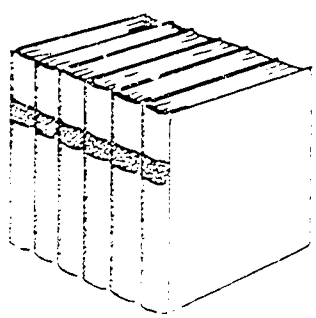
To show c), a slit might be made in one cardboard and another cardboard B could be passed part way through this slit. It is seen that the slit represents a line and that cardboard B can pass through this slit in many different positions. The intersection of plane A and plane B is a line.



Pictures similar to these might be helpful in showing that the intersection of two planes is a line:







INTERSECTIONS OF LINES AND PLANES

Working Together

Do you recall what we mean by the intersection of two sets?

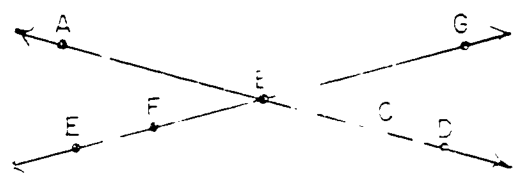
1. Set A = {[illegible]} Set B = {[illegible]}
The intersection of A and B is {[illegible] 8, 9}.

2. Set R = {M, A, R, Y} Set S = {C, A, N, D, Y}
The intersection of R and S is {A, Y}.
The union of R and S is {M, A, R, Y, C, N, D}

3. Set E = {[illegible]} Set F = {9, 10, 11, 12}
The intersection of E and F is { }. (empty set)
The union of E and F is {[illegible] 7, 8, 9, 10, 11, 12}.

You know that a line is a set of points and a plane is a set of points. Let us think about the intersection of two lines.

Look for the points that are on both lines in the picture below.

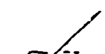


1. What points of $\overleftrightarrow{AC}$ are labeled? of $\overleftrightarrow{EG}$? (A, B, C, D) (E, F, B, G)
2. Is there any point that is on both lines? (Yes)
3. What is the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{EG}$? (Point B)
4. What is the intersection of $\overrightarrow{BA}$ and $\overrightarrow{BG}$? (Point B)
5. What is the intersection of $\overrightarrow{FE}$ and $\overrightarrow{BG}$? (The empty set)

7. How many points are in the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{EF}$? (one)

8. Can you hold two pencils to represent lines so that no point is on both lines? (yes) Can you do this in more than one way? (yes the pencils could be held so they are parallel or so they are skewed)

9. Use a card to represent a plane and a pencil to represent a line. Can you hold [illegible] make their intersection

a) one point? (yes like this or like  for example)
b) no points? (yes, hold them in parallel position)
c) just two points? (no)
d) many points? (yes, hold [illegible] pencil so it is in the plane)

10. Use two cards to represent two planes. Can you hold them [illegible] the intersection of the planes they represent is

a) just one point? [illegible]
b) just two points? (no)
c) more than two points? (yes)
d) no points? (yes, if they are parallel)

11. Which of these pictures show that:

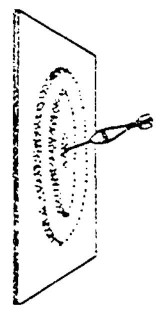
a) the intersection of a line and a plane is one point? (1, 2, 5)

b) the intersection of a line and a plane is a line? (1, 2, 3, 4, 5)

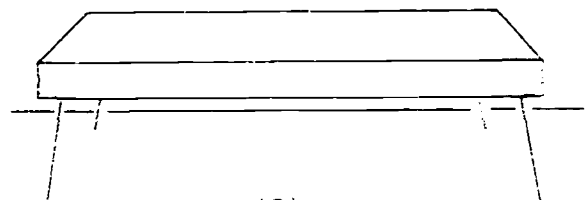
c) the intersection of two planes is a line? (2, 5)

d) the intersection of two lines is a point? (1, 2, 3, 4, 5, 6)

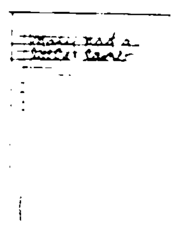
e) the intersection of two lines is the empty set?



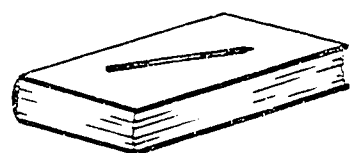
(1)



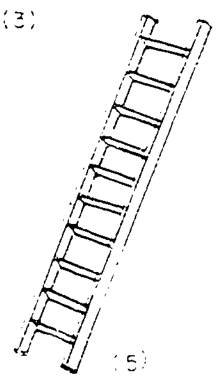
(2)



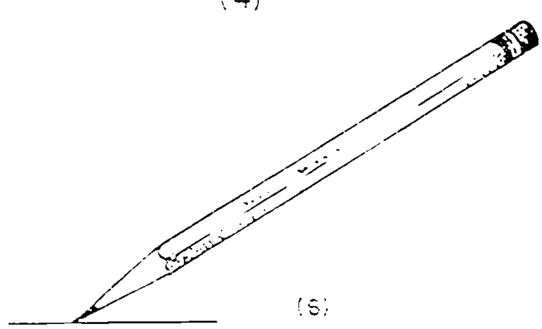
(3)



(4)

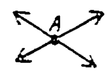


(5)



(6)

Exercise Set 8

1. Mark a point on your paper and label it A. Draw two different lines through A. 
   What is the intersection of the two lines? (point A)

2. Mark two different points on your paper and call them B and C. Can you draw two different lines, both through B and C? (no) Can you draw one line through both B and C? (yes)

3. What word will make this a true sentence?
   If two different lines in a plane cross, their intersection is (one) or (a) ______ point.

4. Can you draw a picture to represent two lines whose intersection seems to be the empty set? (yes) If so, draw it. (lines could be parallel or skewed)

5. Look at the walls, floor, and ceiling of your classroom. Which represent pairs of planes which cross? (a, d)

   a) the side wall and front wall (yes)

   b) the floor and ceiling (no)

   c) the back wall and front wall (no)

   d) the front wall and ceiling (yes)

6. Which of the walls, floor, and ceiling represent planes which do not cross? (b, c)
   a) the floor and side wall
   b) the floor and ceiling
   c) the back wall and front wall
   d) the front wall and ceiling

7. Imagine you have folded a sheet of paper and opened it to form a tent as we did before. Does the folded sheet suggest two intersecting planes? (yes) What is the intersection in this case? (the fold or crease which represents a line.)

8. Complete this sentence. Two intersecting planes in space intersect in a ___?___ (line).

9. If three different points of a line are in a plane, what can you say about the line and the plane? (The whole line is in the plane.)

10. Review
    We have learned the following facts.
    a) If two different lines in a plane cross, their intersection is one point.
    b) If two different planes in space cross, their intersection is one line.
    c) If a line and a plane cross, the intersection is either one point or the entire line.

SIMPLE CLOSED CURVES

Objective: To develop the understanding that a simple closed curve is a path having the following properties:

1. All its points lie in a plane.

2. If one traces the path he eventually returns to the starting point.

3. The path never intersects itself, i.e., in proceeding once around the path any point is encountered just once (except for the starting point.)

Materials: Paper, pencil, chalk.

Vocabulary: closed curve, simple closed curve, boundaries.

Suggested Teaching Procedures:

Ex. 2 in "Working Together" reviews the idea that a line segment is a curve--a special one, of course.

The children's drawings for Ex. 3 will be varied. Perhaps some will have drawn simple closed curves. These might be used as a "stepping stone" to Ex. 4 and 5. The teacher may want to draw on the board a number of closed curves some of which are simple closed curves. The children can then identify the simple closed curves.

The children might enjoy drawing pictures which represent simple closed curves such as:

SIMPLE CLOSED CURVE

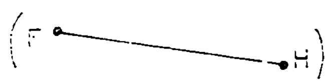
Working Together

In the section on curves, we drew a path from a point A to a point B. We called the set of points the tip of the pencil passed through a curve.

1. Mark two points on your paper and name them T and R. Draw on your paper a picture of a curve from T to R.

(answers will vary)

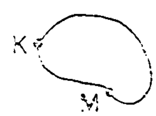
2. Mark two points F and H. Draw $\overline{FH}$. We also call $\overline{FH}$ a curve.



3. a) Mark a point K. Draw a curve that starts at K and comes back to K along a different path. Could you draw the curve using line segments? yes

Since your curve begins and ends at the same point, it is a closed curve.

b) Mark a different point on the curve that contains K and call it M. Can you start at M and trace the curve and come back to M? yes Did you trace every point of the curve? yes



4. Mark a point A and a point B. Draw a curve that begins at A and passes through B and then comes back to A without crossing itself. *(Suggestions for this problem and for number 5 are placed as ex. 8, page 44 in order not to give answer away.)*

Your curve through A and B is called a <u>simple</u> <u>closed</u> <u>curve</u>. It starts at one point and comes back to this point without intersecting itself. All the points of a simple closed curve are in the same plane.

5. Mark a point C and a point D. Draw a curve that starts at C and passes through D twice and then comes back to C.

Your curve through C and D is not a simple closed curve because it intersects itself at D.

6. The curve below does not intersect itself. Why is it <u>not</u> a simple closed curve? *(It is not closed, that is, it does not come back to its starting point.)*

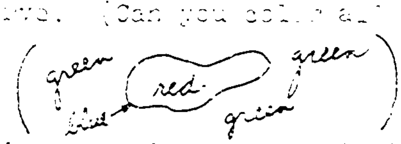
7. Is the figure below a simple closed curve? *(No)* Why? *(It does not start at a point and come back to this point without intersecting itself.)*

[illegible]

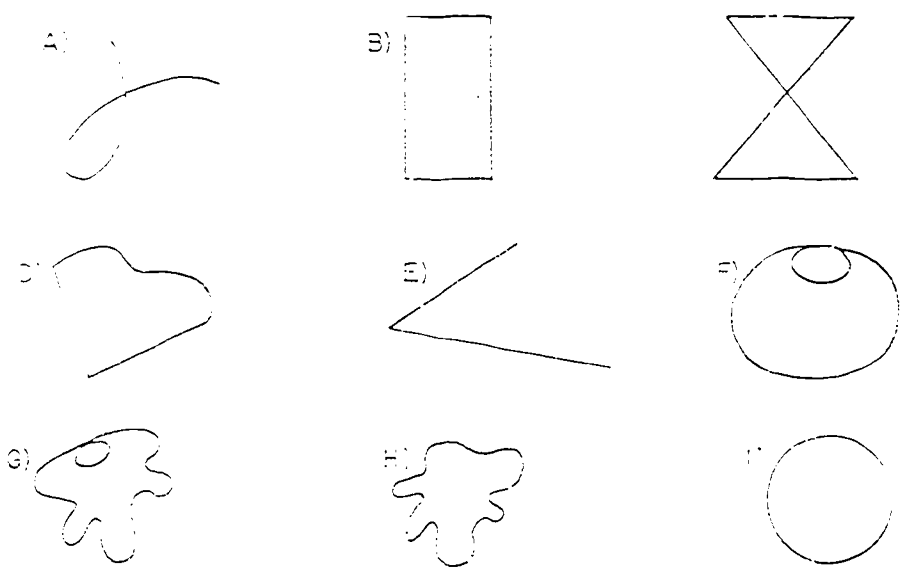
3. A frame around a picture suggests a simple closed curve. Name some other things which suggest simple closed curves. (a fence around a yard, a bicycle tire, etc.)

<u>Exercise Set [illegible]</u>

1. Draw a simple closed curve on your paper. Draw [illegible] with a blue crayon. Color red the part of the [illegible] the inside the curve. Color green the part of the plane outside the curve. (Can you color all [illegible] plane?) (no)



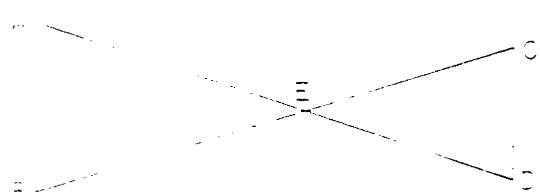
2. Tell which of the following are pictures of simple closed curves. (B, D, H, I)



(c, d, f, g, i, k)

... on the map of the United States ... represent a simple closed curve. ... (For example, Michigan)

... no.



... two

7. Look at the curves in Ex. 3. Give other names for some of the simple closed curves? (a is a triangle; b is a quadrilateral - or a rectangle; c looks like a circle.)

8. Did your curves for Ex. 4 and 5 of page look something like this?

[illegible]

[illegible] simple [illegible] curves. Have them draw some [illegible] simple closed curves and explain why not.

[illegible] introduce the idea that a [illegible] made up of line [illegible]

[illegible]

[illegible] words in the child's [illegible] vocabulary such as [illegible] and [illegible].

[illegible] "Working Together," the teacher [illegible] attention to many objects [illegible] such as [illegible] boxes, picture frames, etc. which have representations of line segments which form polygons.

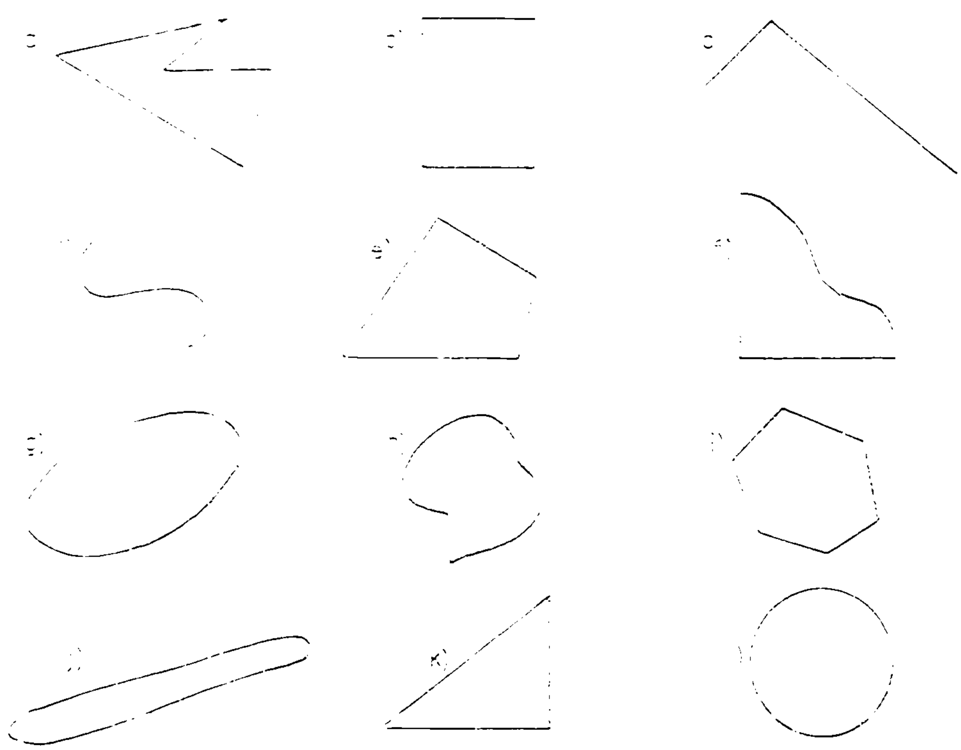
What is a name for your figure? *triangle*

What is a name for your figure? *a quadrilateral*

What is a name for your figure? *a pentagon*

no Why? *Because it does not start at one point and come back to this point without intersecting itself.*

Which of these are pictures of simple closed curves? *(all except b and c)* Which are pictures of polygons? *(a, e, i, k)*



[illegible]

[illegible] *picture [illegible]*

A polygon [illegible]

[illegible]

A polygon [illegible]

[illegible]

[illegible]

[illegible]

[illegible] *b, c*

[illegible]

[illegible]

C• •B

[illegible] a polygon? *yes*

Does the figure represent a triangle? *(yes)*

[illegible] complete this sentence:

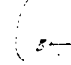
A triangle is made up of *(three)* line segments and these line segments have *(three)* different endpoints.

[illegible]

[illegible] △ B C.

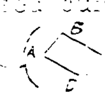
[illegible]
△ BCA or △ ACB
△ CAB or △ CBA

[illegible]

1. a) Draw a picture of a simple closed curve that is the union of three line segments. ( or any other triangle)

b) Label the endpoints of the three line segments. How many different endpoints are there? three

c) What are two names for this kind of simple closed curve? triangle, polygon

2. a) Draw a picture of a simple closed curve that is the union of four line segments. ( or any other quadrilateral)

b) Label the endpoints of the four line segments. How many endpoints are there? four

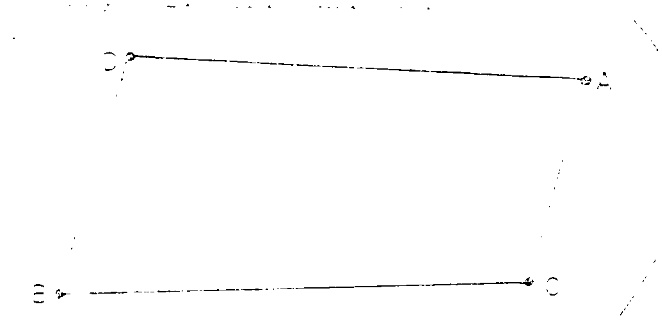
c) What are two names for this kind of simple closed curve? quadrilateral, polygon

3. a) Draw a picture of a simple closed curve that is the union of five line segments. 

b) What is a name for this kind of simple closed curve? polygon (its special name is pentagon)

[illegible] points [illegible]

Dr[illegible] $\overline{[illegible]}$, $\overline{[illegible]}$, $\overline{[illegible]}$, and $\overline{[illegible]}$.



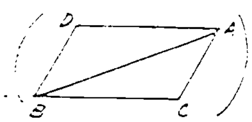
[illegible] of these are names for this figure? (a, b, c,)

a) [illegible] closed curve

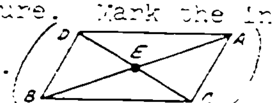
b) polygon

c) triangle

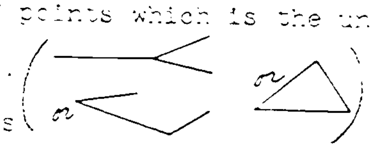
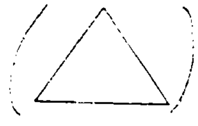
d) quadrilateral

6. Draw $\overline{AB}$ in your drawing for Ex. [illegible] 

How many triangles do you see? (two) Name them. (△BDA, △BCA)

7. Now draw $\overline{DC}$ in the same figure. Mark the intersection of $\overline{AB}$ and $\overline{DC}$. Label it E. 

How many triangles do you see now? (8) Name them.
(△BDA, △BCA, △DAC, △DBC, △BDE, △EDA, △EAC, △ECB)

8. Draw a set of points which is the union of three line segments.  Draw a closed curve which is the union of three line segments.  Can these drawings be different? (yes) Can these drawings be the same? (yes)

[illegible]. Could a polygon have exactly 5,[illegible] sides? *yes*

[illegible]. Could a polygon be the union of two line segments and part of the letter [illegible]? *(no, a polygon is a closed curve which is a union of line segments only.)*

[illegible]. Is the letter O a polygon? *(no, the letter O is not a union of line segments.)*

[illegible]

[illegible] from a point [illegible] all [illegible] equal [illegible].

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] in the [illegible].

[illegible]

1. [illegible] should [illegible] the [illegible].

2. [illegible] the circle.

3. [illegible]

4. [illegible] should be drawn in one [illegible].

Some of the children may become very adept at using a compass. They might want to practice making various designs for room decorations as part of an art project.

F3F.

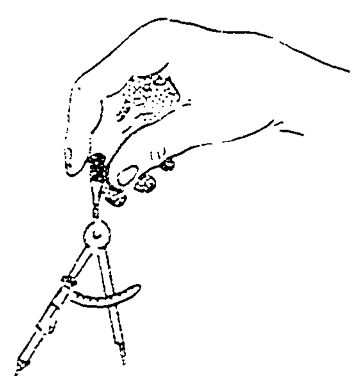
The name of the curve is circle.

A circle cannot be accurately represented by drawing with a pencil and a ruler. A compass is needed.

The easiest way to draw a circle with a compass is to hold the top of the compass between your thumb and index finger. If you press lightly, the compass will work better for you. Press slightly harder on the sharp tip of the compass than you do on the pencil part of the compass.

When you start to draw a circle, do not lift the compass from the paper until the circle is completed. Do not forget to tilt the compass in the direction you are drawing the circle.

Practice using your compass correctly.



[illegible]

[illegible] points in the picture of the simple closed curve you drew? (yes, however answers will vary due to inaccurate drawing or measuring.) If not, change your simple closed curve so that these new [illegible] will be in it.

[illegible] simple closed curve [illegible]? yes

8. Directions for drawing a circle.

a) Mark a point on your paper and label it B. B will not be part of the circle.

b) Set your compass so that the metal tip is two inches from the pencil tip.

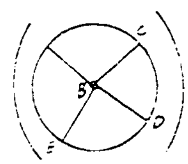
c) Put the metal tip on the point marked B. Now swing the pencil point so that you draw a simple closed curve. Do not let the distance between the pencil point and metal point change while you are drawing.

You have drawn a picture of a circle.

The point marked B is called the <u>center</u> of the circle. Point B is <u>not</u> part of the circle.

9. Mark two points of your circle. Label the points C and D. Draw a picture of $\overline{BC}$.

B marks the center of the circle and C marks a point on the circle.



$\overline{BC}$ is called a <u>radius</u> of the circle.

10. Draw a picture of $\overline{BD}$ in your drawing. $\overline{BD}$ is also a radius of the circle. Why?
(The end points are the center (B) and a point on the circle (D).)

a) Can you draw another radius? (yes) If so, do. Call it $\overline{BE}$.

b) Can you draw still another radius? (yes) If so, do (see above).

c) How many radii does a circle have? (Radii is the plural of radius.) (more than can be counted)

11. Which sentences below are true? Use your picture to help you decide.

a) A circle is all the points the same distance from a point inside it called the center. (true)

b) B names the center of this circle. (true)

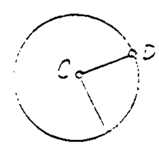
c) All the radii of a circle have the same length. (true)

d) $\overline{BC}$ is a radius of the circle. (true)

e) $\overline{BD}$ and $\overline{BE}$ are both radii of the circle. (true)

<u>Exercise Set [illegible]</u>

1. a) Mark a point on your paper and label it C. Draw a circle with C as center.

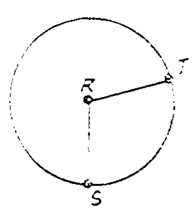


b) Draw a radius of your circle.

c) Mark a point on your circle and label it D. Draw $\overline{CD}$.

d) $\overline{CD}$ is a (radius) of the circle.

2. Look at this picture.



a) Name the center of the circle. (R)

b) Name a radius of the circle. ($\overline{RS}$ and $\overline{RT}$)

c) What is true about the lengths of $\overline{RT}$ and $\overline{RS}$? (they are the same)

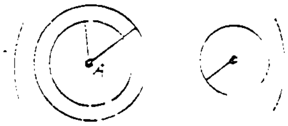
Mark two points about two inches apart. Call the points A and B.

a) Draw a circle with the center at the point A.
(drawings will vary)

b) Draw a different circle with point A as the center.

c) Draw a third circle with point B as the center.

d) Draw a radius of each circle.

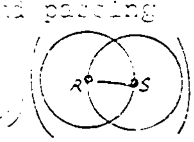


Mark two points R and S about two inches apart.

a) Draw a circle with center at point R and passing through point S.

b) Draw a circle with the center at S and passing through R.

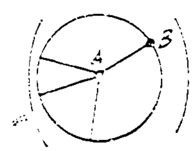
c) Is $\overline{RS}$ a radius of both circles? yes



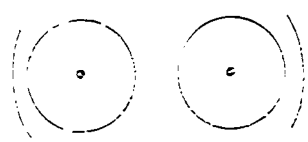
Mark two points A and B on your paper.

a) Draw a circle with the center at A and having $\overline{AB}$ as a radius.

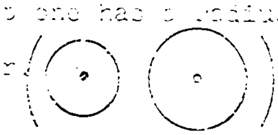
b) Draw three more radii of this circle.



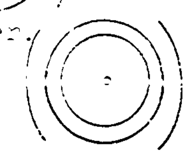
Draw two different circles so that the radius of one has the same length as a radius of the other.



Draw two different circles so that one has a radius of different length from the other.

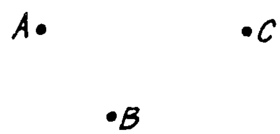


Draw two different circles with the same center.



P285

9. Trace the points of {A, B, C} on your paper.



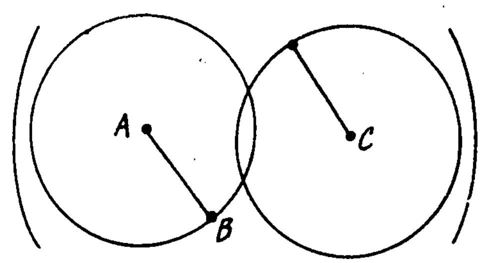
a) Draw $\overline{AB}$.

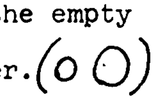
b) Draw the circle with center at A and passing through B.

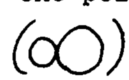
c) Draw a circle with center at C and a rad'
equal in length to the length of $\overline{AB}$.

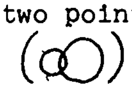
d) Draw a radius of the circle you just made.

e) Is the length of this radius equal to the length of $\overline{AB}$? (yes)



10. a) Could the intersection of two circles be the empty set? (yes) Draw a figure to show your answer. 

b) Could the intersection of two circles be a set with exactly one point? (yes) Draw a figure to show your answer. 

c) Could the intersection of two circles be a set which has exactly two points? (yes) Draw a figure to show your answer. 

11. a) Could the intersection of a circle and a line be the empty set? (Yes) Draw a figure to show your answer. ()

b) Could the intersection of a circle and a line be a set which has exactly one point? (Yes) Draw a figure to show your answer. ()

c) Could the intersection of a circle and a line be a set which has exactly two points? (Yes) Draw a figure to show your answer. ()

BRAINTWISTER

12. a) Could the intersection of two circles be a set which has exactly three points? (No. If the circles have three points in common then the circles are identical and hence they have all their points in common.)

b) Could the intersection of a circle and a line be a set which has exactly 3 points? (No)

REGIONS IN A PLANE

Objective: To develop the idea of interior and exterior of a simple closed curve; to develop the understanding that the union of a simple closed curve and its interior is called a region.

Materials: Paper, pencil, ruler, compass, chalk.

Vocabulary: region, interior, exterior, triangular region, circular region.

Suggested Teaching Procedures:

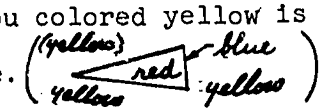
> The development in the pupils' text is in sufficient detail to be followed:
>
> Ex. 1 - 7 develop the idea of <u>interior</u> and <u>exterior</u>. In Ex. 1 - 3, we are <u>really talking</u> about <u>three</u> sets of points--the triangle is a set of points, its interior is a set of points, and its exterior is a set of points. In Ex. 4 - 6 concerning the circle, we are again talking about three sets of points. Ex. 7 checks the pupils' understanding of these three sets of points.
>
> Ex. 8 and 9 develop the understanding that the union of a simple closed curve and its interior is called a <u>region</u>.
>
> In Ex. 2b, Exercise Set 10, children may not be able to answer independently that this is named a <u>circular region</u>. It is hoped that some children <u>will be able to</u> do so independently. Ex. 8, 9, and 11 use the term, <u>circular region</u>.

REGIONS IN A PLANE

Working Together

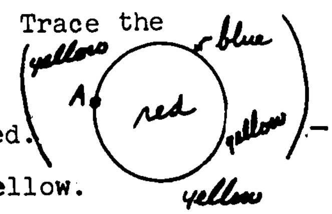
1. Draw a picture of a triangle. Trace the triangle with a blue crayon.

2. Color the part of the plane inside the triangle red. The set of points you colored red is called the interior of the triangle.

3. Color the part of the sheet outside the triangle yellow. This set of points which you colored yellow is part of the exterior of the triangle. 

The set of points of the triangle is not in the interior and is not in the exterior of the triangle.

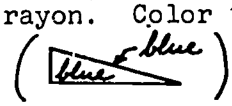
4. Use your compass to draw a circle. Trace the circle with a blue crayon.



5 Color the interior of the circle red.

6. Color the exterior of the circle yellow.

7. Mark a point of the circle. Label it A. Is point A in the interior of the circle? (no) Is A in the exterior of the circle? (no) Mark another point which is not in the interior of the circle and is not in the exterior of the circle. 

8. Draw a triangle with blue crayon. Color the interior of the triangle blue also. 

9. The part of the plane colored blue is the union of two sets of points. What two sets? (The set pictured by the triangle and the set pictured by the interior of the triangle)

The union of a simple closed curve and its interior is called a plane region. The one you colored blue is called a triangular region.

Exercise Set 10

1. a) Draw a triangle. Color the triangle and its interior red.

   b) What is the name of the part of the plane which is red? (triangular region)

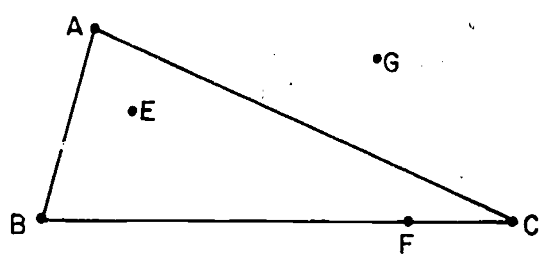
   c) What is the name of the part of the plane which is not red? (the exterior of the triangle)

2. a) Draw a circle. Color the circle and its interior blue.

   b) What do you think should be the name of the part of the plane which is blue? (circular region)

   c) What is the name of the part of the plane which is not blue? (the exterior of the circle)

3. Look at the figure and the labeled points.



Which sentences are true? (b, d)

E is: a) a point of the triangle.

b) a point of the interior of the triangle.

c) a point of the exterior of the triangle.

d) a point of the triangular region.

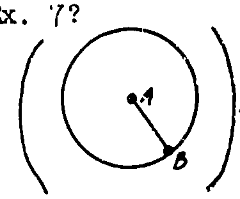
4. F is: a) a point of the triangle. (a, d)
   b) a point of the interior of the triangle.
   c) a point of the exterior of the triangle.
   d) a point of the triangular region.

5. G is: a) a point of the triangle. (c)
   b) a point of the interior of the triangle.
   c) a point of the exterior of the triangle.
   d) a point of the triangular region.

6. A is: a) a point of the triangle. (a, d)
   b) a point of the interior of the triangle.
   c) a point of the exterior of the triangle.
   d) a point of the triangular region.

7. Mark a point A and a point B at least two inches from A. Draw a circle with center A and with $\overline{AB}$ a radius.

Which endings are correct for the figure in Ex. 7?



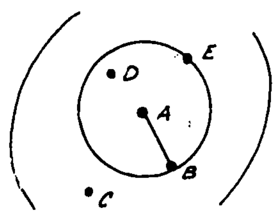
8. A is a point of (b, d)
   a) the circle.
   b) the interior of the circle.
   c) the exterior of the circle.
   d) the circular region.

9. B is a point of (a, d)
   a) the circle.
   b) the interior of the circle.
   c) the exterior of the circle.
   d) the circular region.

10. Mark a point of the exterior of your circle. Label it C.

11. Mark a point of the circular region. Label it D.

12. Mark a point which is not in the interior and not in the exterior of the circle. Label it E.



ANGLES

Objective: To develop the understanding that an angle is a set of points satisfying the following conditions:

1. It is the union of two rays with a common endpoint.
2. The rays do not lie on the same line.
3. The common endpoint of the rays is called the vertex. It is also the only member of the intersection of the rays.
4. Each ray is called a side of the angle.

Materials: Paper, pencil, ruler, chalk, chalkboard, styrofoam, toothpicks, compass.

Vocabulary: angle, vertex.

Suggested Teaching Procedures:

The teacher points out parts of angles represented by various objects he shows his class. He can ask the children to point out other examples of parts of angles they can see in their classroom. These may be parts of angles represented by edges of their desks which meet, line segments which meet at the corner of the room, the edges of their books which meet, the different angles their compasses partially represent when the opening is changed. Children may recall angles they have seen represented in bridge structures and in street intersections. They may think about the angle suggested by the cross-bars of a telephone pole; they may point out the angles partially represented in some letters of the alphabet. The teacher or the children may make pictorial representations of these examples on the board or with toothpicks stuck in styrofoam. At this time the teacher may find it helpful to conduct a lesson in paper folding to illustrate the representation of

angles. She should give the children an opportunity to label parts of an angle with the correct terms, i.e., angle, vertex, ray.

The "Working Together" section might then be done together as a class activity.

In Ex. 2 of "Working Together" the second ray should not be drawn on line RS. If it were, this would make both rays on the same line and an angle, as we have defined it, would not be formed.

Ex. 6 gives the symbol for angle and indicates how angles are to be named.

When drawing representations of angles, it might be a good idea to have them in various positions so that pupils do not get the notion that one of the rays of an angle must be parallel to the bottom of their paper.

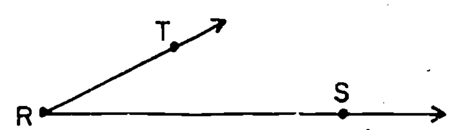
ANGLES

Working Together

1. Mark a point R on your paper. Draw a ray with R as endpoint. Mark another point on the ray and label it S.

2. Draw a second ray with R as endpoint. Do not draw it on $\overleftrightarrow{RS}$. Mark a point on this ray and label it T.

Does your drawing look something like this?



This drawing represents a new geometric figure called an angle.

An angle is the union of two rays which have the same endpoint but are not on the same line.

In the figure, R is the vertex of the angle. The endpoint of both rays is called the vertex of the angle.

Each ray is a <u>ray of the angle</u>. $\overrightarrow{RT}$ and $\overrightarrow{RS}$ are rays of the angle in the drawing.

3. Part of an angle is represented by two edges of your desk which meet at a corner.

a) What represents the vertex of the angle? *(the corner - the point where the edges meet)*

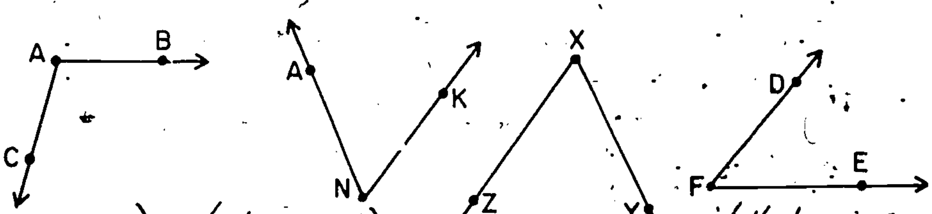
b) What represents the rays of the angle? *(the edges of the desk top)*

c) Why do we say these are only part of the angle? *(the rays extend without ending. the edges of the desk top end)*

4. Do the hands of a clock suggest an angle? *(yes)* If so, what represents the vertex? What represents the rays? *(the center of the clock where the hands join, represents the vertex)* *(the hands of the clock represent the rays)*

5. Describe other things in your classroom which suggest an angle. *(answers will vary)*

6. In each angle pictured below, name the vertex and the rays.

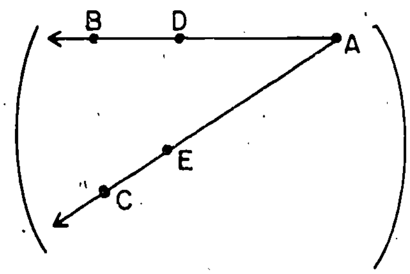


(Vertex is A, rays are $\overrightarrow{AB}$ and $\overrightarrow{AC}$) *(Vertex is N, rays are $\overrightarrow{NA}$ and $\overrightarrow{NK}$)* *(Vertex is X, rays are $\overrightarrow{XZ}$ and $\overrightarrow{XY}$)* *(Vertex is F, rays are $\overrightarrow{FD}$ and $\overrightarrow{FE}$)*

We name the first angle in the picture $\angle BAC$ or $\angle CAB$. Either is correct. The middle letter must be the label for the vertex.

7. Draw an angle. Label it $\angle SRT$. Did you put the correct letter at the vertex? (R)

8. Below is represented $\angle BAC$. Copy the picture on your paper.



a) Choose a point on $\overrightarrow{AB}$ different from A and B and label it D.

b) Choose a point on $\overrightarrow{AC}$ different from A and C and label it E.

c) Is $\overrightarrow{AB}$ the same ray as $\overrightarrow{AD}$? (yes)

d) Is $\overrightarrow{AC}$ the same ray as $\overrightarrow{AE}$? (yes)

e) Is $\angle BAC$ the same angle as $\angle DAE$? (yes)

No matter how we label an angle, the middle letter always represents the vertex.

9. Three points are shown below.

Write on a sheet of paper the words that complete these sentences.

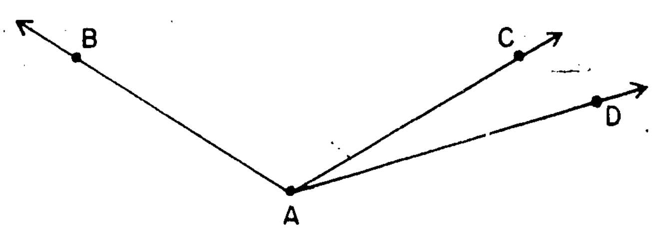
a) There is (one) ___?___ ray through D and F with endpoint F.

b) There is (one) ___?___ ray through F and E with endpoint F.

c) There is (one) ___?___ angle containing D and E with vertex F. This angle is labeled ($\angle DFE$) ___?___ or ($\angle EFD$) ___?___.

<u>Exercise Set 11</u>

1. Here are three rays. Each has the endpoint A. Name three angles.



($\angle BAD$ or $\angle DAB$
$\angle BAC$ or $\angle CAB$
$\angle CAD$ or $\angle DAC$)

2. a) Mark a point C on your paper. Draw a picture of two angles which have the point marked C as a vertex.

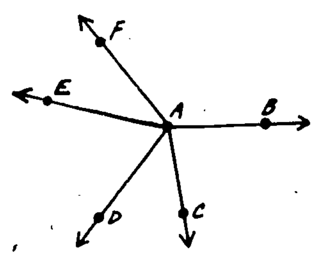
   b) Name the rays of each angle.

   ($\overrightarrow{CB}$ and $\overrightarrow{CD}$, $\overrightarrow{CD}$ and $\overrightarrow{CE}$, $\overrightarrow{CB}$ and $\overrightarrow{CE}$).

3. a) Mark a point A on your paper. Draw a picture of at least 4 angles which have the point marked A as a vertex. Do this by drawing 5 different rays, not on the same line, with A as endpoint. Choose a point different from A on each ray. Label these points with the capital letters B, C, D, E, and F.

   b) Name the rays of each angle.

   c) Name each angle.

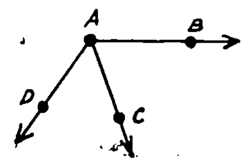


$\angle BAC \begin{cases}\overrightarrow{AB}\\ \overrightarrow{AC}\end{cases}$, $\angle CAD \begin{cases}\overrightarrow{AC}\\ \overrightarrow{AD}\end{cases}$, $\angle DAE \begin{matrix}\overrightarrow{AD}\\ \overrightarrow{AE}\end{matrix}$, $\angle EAF \begin{cases}\overrightarrow{AE}\\ \overrightarrow{AF}\end{cases}$

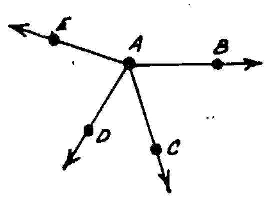
also $\angle BAD$, $\angle BAF$, $\angle CAE$, $\angle BAE$, $\angle CAF$, $\angle DAF$

BRAINTWISTERS

4. Try to repeat Ex. 3 by using only 3 rays (no two of them on the same line) with A as endpoint. Did you get a picture of four angles? (no) How many angles does your picture represent?

(3, since  shows ∠BAC, ∠CAD and ∠BAD)

5. Try to repeat Ex. 3 by using only 4 rays (no two of them on the same line) with A as endpoint. Did you get a picture of at least four angles? (Yes) Did you get a picture of exactly four angles? (No) How many angles does your picture represent?

(6, since  shows ∠BAC, ∠CAD, ∠DAE, ∠BAD, ∠CAE, and ∠BAE.)

ANGLES OF A TRIANGLE

Objective: To develop the understanding that even though a triangle determines three angles, these angles have some points which are not a part of the triangle. This is true because angles are formed by the union of rays while triangles are formed by the union of line segments.

Materials: Paper, pencil, ruler, chalk.

Vocabulary: vertices.

Suggested Teaching Procedures:

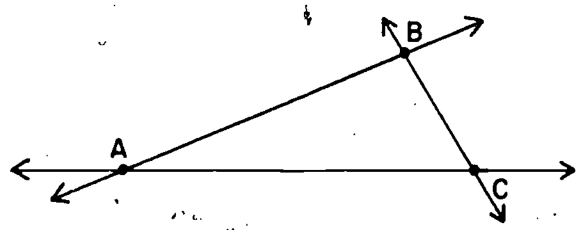
The objective as stated above is a difficult one to comprehend. The "Working Together" section is designed to reach this objective through a series of activities. Best results might be obtained by following the procedure as given in the pupils' text.

You may want to go over the "Working Together" section using the chalkboard and with the pupils' text closed.

ANGLES OF A TRIANGLE

Working Together

1. Look at the points below which are labeled A, B, and C. They are not on the same line. Mark three points like this on your paper and label them.



2. Draw: $\overrightarrow{AB}$

$\overrightarrow{AC}$

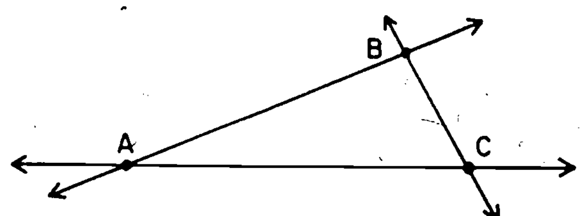
$\overrightarrow{BC}$

$\overrightarrow{BA}$

$\overrightarrow{CB}$

$\overrightarrow{CA}$

3. Does your drawing look something like this? (yes)



Write on a sheet of paper the words that complete these sentences.

a) $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ form a ____?____. (triangle)

b) The angle with vertex A which contains B and C is called ____?____. ($\angle BAC$ or $\angle CAB$)

c) The angle with vertex B which contains A and C is called ____?____. ($\angle ABC$ or $\angle CBA$)

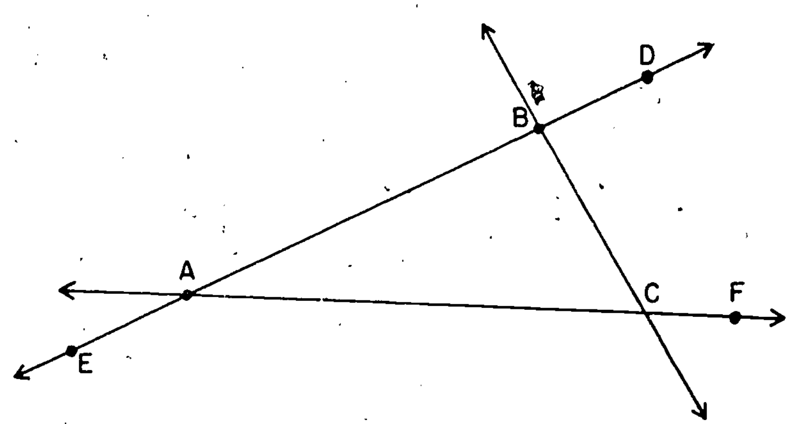
d) The angle with vertex C which contains A and B is called ____?____. ($\angle ACB$ or $\angle BCA$)

4. Mark a point of $\overrightarrow{AB}$ which is not a point of $\overline{AB}$. Label it D.

5. Mark a point of $\overrightarrow{BA}$ which is not a point of $\overline{AB}$. Label it E.

6. Mark a point of $\overrightarrow{AC}$ which is not a point of $\overline{AC}$. Label it F.

Does your drawing look like this now? (yes)



7. Are D, E, and F points of the rays of the angles you named in Ex. 3? (yes)

8. a) Are D, E, and F points of the triangle? (no)

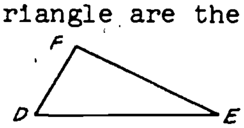
   b) Are D, E, and F points of the interior of the triangle? (no)

   c) Are D, E, and F points of the exterior of the triangle? (yes)

Ex. 3 shows that a triangle suggests three angles. These angles are not part of the triangle. This is true because a triangle is made up of segments and an angle is made up of rays.

Remember when we studied circles we spoke of the center of a circle. The center is not part of the circle.

In the same way we say ∠ABC, ∠BCA, and ∠CAB are angles of the triangle although they are not part of the triangle. We call the vertices of these angles the vertices of the triangle. <u>Vertices</u> is the plural of vertex. The vertices of a triangle are the endpoints of the segments of the triangle.



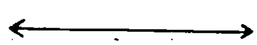
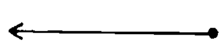
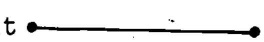
9. Draw a triangle. Label its vertices D, E, F.

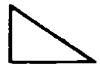
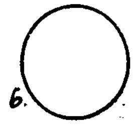
a) Name the three angles of the triangle (∠DFE, ∠FDE, ∠DEF)

b) The three angles of a triangle suggest how many rays? (six)

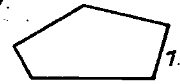
Exercise Set 12

Make drawings to represent

1. A line 
2. A ray 
3. A segment 
4. A simple closed curve 

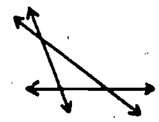
5. A triangle  

6. A circle

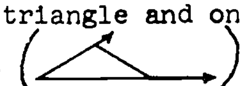
7. A polygon  or any other polygon

8. Two lines which cross 

9. A quadrilateral 

10. Three lines which cross but not all in the same point 

11. An angle 

12. The union of a triangle and one angle suggested by the triangle 

13. A triangular region 

Using the drawing below name:

14. the intersection of $\overrightarrow{AB}$ and $\overrightarrow{DC}$. (point B)

15. three different triangles. (△ABC, △ACD, △ABD)

16. a segment which is not a side of a triangle. ($\overline{DF}$ or $\overline{AE}$ or $\overline{EC}$)

17. a point of the interior of some triangle. (E in the interior of △ABD)

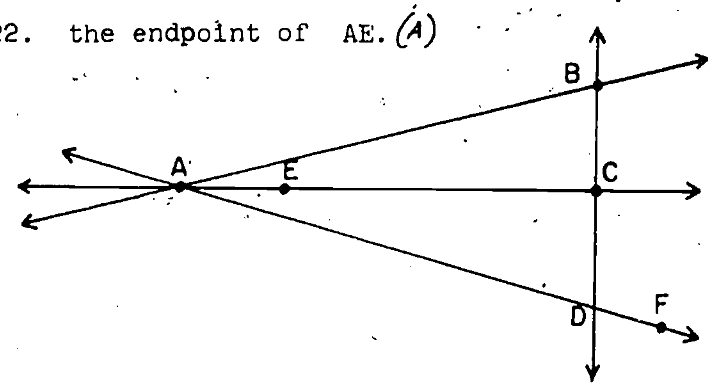
18. a point of the exterior of triangle ABD. (F)

19. the intersection of $\overleftrightarrow{AF}$ and $\overleftrightarrow{BC}$. (point D)

20. the intersection of $\overleftrightarrow{AC}$ and $\overrightarrow{DF}$. (the empty set: they do not cross)

21. the intersection of $\overline{AE}$ and $\overline{BD}$. (the empty set: they do not cross)

22. the endpoint of AE. (A)



Which sentences are true?

23. The intersection of two different planes may be: (a, b)
    a) a line.
    b) the empty set.
    c) a set which has exactly one point.
    d) a plane.

24. The intersection of a line and a plane may be: (b, c, e)
    a) a set which has exactly two points.
    b) the empty set.
    c) the line.
    d) the plane.
    e) a set which has exactly one point.

BRAINTWISTERS

25. The intersection of a triangle and a plane may be: (a, b, c, e)
    a) a set which has exactly one point.
    b) the empty set.
    c) the triangle.
    d) a set which has exactly three points.
    e) a set which has more points of the triangle than can be counted but not all the points of the triangle.

26. The intersection of a circle and a plane may be: (a, b, c, e)
    a) a set which has exactly one point.
    b) the empty set.
    c) a set which has exactly two points.
    d) a set which has exactly three points.
    e) the circle.